On the weakly modular form of weight k

Sachin Kumar
Faculty of Mathematics, Univeristy of Waterloo

April 2024

Abstract

Modular forms play a super important role in number theory. In fact, the Modularity Theorem, which asserts that elliptic curves over the rationals are related to modular forms in a particular way, implies Fermat's Last Theorem! We are familiar with the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ with $\mathfrak{R}(s) > 1$, and using the Summation Theorem, we recall that $\zeta(2) = \frac{\pi^2}{6}$. In this essay, we will analyze its two-dimensional analogue.

Contents

1 Definition

Definition 1.1 (weakly modular form of weight k). Let $\mathcal{H} = \{z \in \mathbb{C} : \mathfrak{I}(z) > 0\}$ denote the upper half-plane. We say $f : \mathcal{H} \to \mathbb{C}$ is a ***weakly modular form of weight*** k, where $k \in \mathbb{N}$ if and only if

(a) $f(\tau + 1) = f(\tau)$ for every $\tau \in \mathcal{H}$ (i.e., f is periodic with period 1)

(b) $f\left(\frac{-1}{\tau}\right) = \tau^k f(\tau)$ for every $\tau \in \mathcal{H}$

(c) f is analytic in $\mathcal{H}$

(d) f is analytic at ∞

We nore that in this context, we say that f is analytic at ∞ if and only if there exist positive real numbers R and M such that f is analytic at τ and $|f(\tau)| \leq M$ whenever $\mathfrak{I}(\tau) \geq R$.

We are familiar with the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ with $\mathfrak{R}(s) > 1$, and using the Summation Theorem, we recall that $\zeta(2) = \frac{\pi^2}{6}$. Let's now analytic its two-dimensional analogue

$$G_k : \mathcal{H} \to \mathbb{C}, \;\; G_k(\tau) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(m\tau + n)^k}$$

where $k > 2$ is an integer.

2 Our Claims

LEMMA 2.1. *G_k is a weakly modular form of weight k.*

Before providing the proof, we will dicuss the idea behind the proof. First, we will show that the series,

$$\sum_{\substack{(c,d)\in\mathbb{Z}^2\\(c,d)\neq(0,0)}} \frac{1}{\max\{|c|,|d|\}^k}$$

converges. Then using this fact, we prove that $G_k(\tau)$ converges absolutely and uniformly for all $\tau \in \Omega_{A,B}$, with A and B are positive real numbers, where

$$\Omega_{A,B} = \{\tau \in \mathcal{H} \,:\, |\Re(\tau)| \le A \text{ and } \Im(\tau) \ge B\}$$

Let's provide the proof,

PROOF (LEMMA 2.1). To show that G_k is a weakly modular form of weight k, we will have to show that G_k satisfies the conditions in Definition 1.1,

(a) We observe that,

$$\begin{aligned} G_k(\tau+1) &= \sum_{\substack{(m,n)\in\mathbb{Z}^2\\(m,n)\neq(0,0)}} \frac{1}{(m(\tau+1)+n)^k} = \sum_{\substack{(m,n)\in\mathbb{Z}^2\\(m,n)\neq(0,0)}} \frac{1}{(m\tau+(m+n))^k} \\ &= \sum_{\substack{(m,n')\in\mathbb{Z}^2\\(m,n'-m)\neq(0,0)}} \frac{1}{(m\tau+n')^k} \qquad \text{where } n' = m+n \\ &= \sum_{\substack{(m,n')\in\mathbb{Z}^2\\(m,n')\neq(0,0)}} \frac{1}{(m\tau+n')^k} = G_k \end{aligned}$$

Since, we know that $(m, n'-m) = (0,0)$ if and only if $(m,n') = (0,0)$.

(b) We observe that,

$$\begin{aligned} G_k\left(\frac{-1}{\tau}\right) &= \sum_{\substack{(m,n)\in\mathbb{Z}^2\\(m,n)\neq(0,0)}} \frac{1}{\left(m\left(\frac{-1}{\tau}\right)+n\right)^k} = \tau^k \sum_{\substack{(m,n)\in\mathbb{Z}^2\\(m,n)\neq(0,0)}} \frac{1}{(n\tau-m)^k} \\ &= \tau^k \sum_{\substack{(m',n')\in\mathbb{Z}^2\\(-n',m')\neq(0,0)}} \frac{1}{(m'\tau+n')^k} \qquad \text{where } m' = n \text{ and } n' = -m \\ &= \tau^k \sum_{\substack{(m',n')\in\mathbb{Z}^2\\(m',n')\neq(0,0)}} \frac{1}{(m\tau+n')^k} = \tau^k G_k \end{aligned}$$

Since, we know that $(-n', m') = (0,0)$ if and only if $(m', n') = (0,0)$.

(c) Let us show that the series,

$$\sum_{\substack{(c,d)\in\mathbb{Z}^2\\(c,d)\neq(0,0)}} \frac{1}{\max\{|c|,|d|\}^k}$$

converges. We have

$$\begin{aligned}
\sum_{\substack{(c,d)\in\mathbb{Z}^2\\(c,d)\neq(0,0)}} \frac{1}{\max\{|c|,|d|\}^k} &= \sum_{\substack{d\in\mathbb{Z}\\d\neq 0}} \frac{1}{\max\{0,|d|\}^k} + \sum_{\substack{c\in\mathbb{Z}\\c\neq 0}}\sum_{d\in\mathbb{Z}} \frac{1}{\max\{|c|,|d|\}^k}\\
&= 2\sum_{d=1}^{\infty}\frac{1}{d^k} + 2\sum_{c=1}^{\infty}\sum_{d\in\mathbb{Z}}\frac{1}{\max\{c,|d|\}^k}\\
&= 2\sum_{d=1}^{\infty}\frac{1}{d^k} + 2\sum_{c=1}^{\infty}\frac{1}{c^k} + 2\sum_{c=1}^{\infty}\sum_{\substack{d\in\mathbb{Z}\\d\neq 0}}\frac{1}{\max\{c,|d|\}^k}\\
&= 4\zeta(k) + 4\sum_{c=1}^{\infty}\sum_{d=1}^{\infty}\frac{1}{\max\{c,d\}^k}\\
&= 4\zeta(k) + 4\sum_{c=1}^{\infty}\left(\sum_{d=1}^{c}\frac{1}{c^k} + \sum_{d=c+1}^{\infty}\frac{1}{d^k}\right)\\
&= 4\zeta(k) + 4\sum_{c=1}^{\infty}\left(\frac{1}{c^{k-1}} + \sum_{d=c+1}^{\infty}\frac{1}{d^k}\right)\\
&= 4\zeta(k) + 4\zeta(k-1) + 4\sum_{c=1}^{\infty}\sum_{d=c+1}^{\infty}\frac{1}{d^k}
\end{aligned}$$

At this point we apply the integral test:

$$\begin{aligned}
\sum_{\substack{(c,d)\in\mathbb{Z}^2\\(c,d)\neq(0,0)}} \frac{1}{\max\{|c|,|d|\}^k} &\leq 4\zeta(k) + 4\zeta(k-1) + 4\sum_{c=1}^{\infty}\sum_{d=c+1}^{\infty}\frac{1}{d^k}\\
&\leq 4\zeta(k) + 4\zeta(k-1) + 4\sum_{c=1}^{\infty}\left(\frac{1}{(c+1)^k} + \int_{c+1}^{\infty}\frac{1}{x^k}\,dx\right)\\
&= 4\zeta(k) + 4\zeta(k-1) + 4(\zeta(k)-1) + 4\sum_{c=1}^{\infty}\frac{1}{(k-1)(c+1)^{k-1}}\\
&= 4\zeta(k) + 4\zeta(k-1) + 4(\zeta(k)-1) + 4\sum_{c=1}^{\infty}\frac{1}{(c+1)^{k-1}}\\
&\leq 4\zeta(k) + 4\zeta(k-1) + 4(\zeta(k)-1) + 4(\zeta(k-1)-1)
\end{aligned}$$

Hence the series converges absolutely.
Now, consider the set $\Omega_{A,B}$. We will claim (w/o proof) that there exists a constant $C > 0$, which depends only on A and B, such that $|\tau + \delta| > C\max\{1, |\delta|\}$ for every

$\tau \in \Omega_{A,B}$ and every $\delta \in \mathbb{R}$. Consequently,

$$\begin{aligned}
\left| \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^k} \right| &\leq \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{|m\tau+n|^k} \\
&= 2\zeta(k) + \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ m\neq 0}} \frac{1}{|m|^k \left|\tau + \frac{n}{m}\right|^k} \\
&< 2\zeta(k) + \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ m\neq 0}} \frac{1}{|m|^k C^k \max\left\{1, \left|\frac{n}{m}\right|\right\}^k} \\
&< 2\zeta(k) + \frac{1}{C^k} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ m\neq 0}} \frac{1}{\max\{|m|,|n|\}^k} \\
&\leq 2\zeta(k) + \frac{1}{C^k} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)\neq(0,0)}} \frac{1}{\max\{|c|,|d|\}^k}
\end{aligned}$$

Hence the series converges absolutely and uniformly on $\Omega_{A,B}$. Since for every $\tau \in \mathcal{H}$ there exists $A, B \in \mathbb{R}_+$ such that $\tau \in \Omega_{A,B}$, it follows from the Analytic Convergence Theorem that G_k is analytic in $\mathcal{H}$.

(d) Let $R = 1$ and let

$$M = 2\zeta(k) + \frac{1}{C^k} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)\neq(0,0)}} \frac{1}{\max\{|c|,|d|\}^k}$$

From Part (c), we know that $G_k(\tau)$ analytic in $\Omega_{1,1}$. It remains to prove that $|f(\tau)| \leq M$ whenever $\Im(\tau) \geq R$. Suppose that τ satisfies $\Im(\tau) \geq R$. We consider two cases.
Case 1. Suppose that $|\Re(\tau)| \leq 1$. Then $\tau \in \Omega_{1,1}$ and from Part (c), we know that $G_k(\tau)$ is bounded on $\Omega_{1,1}$ by M. So the result holds in this case.
Case 2. Suppose that $|\Re(\tau)| > 1$. It follows from Part (a) that $G_k(\tau) = G_k(\tau + \ell)$ for every integer ℓ. We choose ℓ so that $0 \leq \Re(\tau) + \ell < 1$. Since $\Re(\tau + \ell) = \Re(\tau) + \ell$ and $\Im(\tau + \ell) = \Im(\tau)$, we conclude that $\tau + \ell \in \Omega_{1,1}$. But then $|G_k(\tau + \ell)| \leq M$, and so $|G_k(\tau)| = |G_k(\tau + \ell)| \leq M$, as expected. Hence G_k is analytic at ∞.

Combining (a), (b), (c), (d), we conclude that G_k is a weakly modular form of weight k. □

LEMMA 2.2. *For even $k > 2$, the Fourier Series of $E_k(\tau) = G_k(\tau)/2\zeta(k)$ is given by*

$$E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi i n \tau}$$

Here B_k is the k^{th} Bernoulli number and $\sigma_m = \mathbb{N} \to \mathbb{N}$ is the m^{th} sum of positive divisor function:

$$\sigma_m(n) = \sum_{\substack{d|n \\ d>0}} d^m$$

Perhaps surprisingly, the set of all weakly modular forms of weight k, denoted $\mathcal{M}_k$, forms a finite-dimensional vector space over $\mathbb{C}$. Use the fact that the vector space $\mathcal{M}_8$ is one-dimensional, as well as the fact that the Fourier series is unique, to prove that $E_8(\tau) = (E_4(\tau))^2$ for every $\tau \in \mathcal{H}$.

PROOF (LEMMA 2.2). We observe that, since $E_4(\tau)$ is a weakly modular form of weight 4, and $(E_4(\tau))^2$ is a weakly modular form of weight 8. Using the similar technique used in Lemma 2.1, we can verify that all the desired four properties are satisfied. Now, since the vector space $\mathcal{M}_8$ is one-dimensional and $E_8, E_4^2 \in \mathcal{M}_8$, the set $\{E_8, E_4^2\}$ is linearly dependent. Since neither E_8 nor E_4^2 are identically equal to zero, it must be the case that E_4^2 is a scalar multiple of E_8, i.e., $E_8(\tau) = c(E_4(\tau))^2$ for some nonzero $c \in \mathbb{C}$. But the leading coefficient of E_8 is 1, while the leading coefficient of $c(E_4(\tau))^2$ is c. Since the Fourier series is unique, the leading coefficients must be equal, and so we conclude that $c = 1$. Hence, $E_8 = E_4^2$. □

LEMMA 2.3. *By equating the coefficients in the relation $E_8(\tau) = (E_4(\tau))^2$, we have*

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{i=1}^{n-1} \sigma_3(i)\sigma_3(n-i)$$

for every $n \in \mathbb{N}$.

PROOF (LEMMA 2.3). We recall that if $\sum_{a_i q^i}$ is a formal power series, then the n^{th} coefficient of $\sum b_k q^k = (\sum a_i q^i)^2$ is equal to

$$b_n = \sum_{i=0}^{n} a_i a_{n-i}$$

Now, let $q = e^{2\pi i\tau}$. From Lemma 2.2 we know that

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) e^{2\pi i n\tau} = \sum_{n=0}^{\infty} a_n q^n$$
$$E_8(\tau) = 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n) e^{2\pi i n\tau} = \sum_{n=0}^{\infty} b_n q^n$$

where

$$a_0 = 1,\ a_n = 240\sigma_3(n)$$
$$b_0 = 1,\ b_n = 480\sigma_7(n)$$

for $n \in \mathbb{N}$. Since, $E_8 = E_4^2$ and the Fourier series is unique, we conclude that, for every

$n \in \mathbb{N}$,

$$\begin{aligned}
480\sigma_7(n) &= b_n \\
&= \sum_{i=0}^{n} a_i a_{n-i} \\
&= 2a_0 a_n + \sum_{i=1}^{n-1} a_i a_{n-i} \\
&= 480\sigma_3(n) + \sum_{i=1}^{n-1} (240\sigma_3(i))(240\sigma_3(n-i)) \\
&= 480\left(\sigma_3(n) + 120\sum_{i=1}^{n-1} \sigma_3(i)\sigma_3(n-i)\right)
\end{aligned}$$

dividing both sides by 480, we get our desired identity

$$\sigma_7(n) = \sigma_3(n) + 120\sum_{i=1}^{n-1} \sigma_3(i)\sigma_3(n-i)$$

Hence proved. □