Solving quadratic equation without a formula!

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This essay will discuss about an ancient Babylonian method to solve the quadratic equation, which got buried after the introduction of the quadratic formula. But why is this a better method? In trivial equations, this method may look tedious but for non-trivial equations, this method is extremely effective and requires basic calculations.

Traditional method: Quadratic formula

Let $f(x)$ be a quadratic equation of the form,

$$\alpha x^2 + \beta x + \gamma = 0$$

with $\alpha, \beta, \gamma \in \mathbb{R} \setminus 0$. Then,

$$x = \frac{-\beta \pm \sqrt{d}}{2\alpha}$$

where $d = \beta^2 - 4\alpha\gamma$, is called the discriminant.

- When $d > 0$, there exists $x_1, x_2 \in \mathbb{R}$, where $x_1 \neq x_2$.
- When $d = 0$, there exists $x_1, x_2 \in \mathbb{R}$, where $x_1 = x_2$.
- When $d < 0$, there exists $x_1, x_2 \in \mathbb{C}$, where $x_1 \neq x_2$ and are complex conjugates of each other.

As a typical mathematician, let's derive the quadratic formula!
Let's start with $\alpha x^2 + \beta x + \gamma = 0$, then dividing by α, we get

$$x^2 + \frac{\beta}{\alpha}x + \frac{\gamma}{\alpha} = 0$$

$$x^2 + \frac{\beta}{\alpha}x = -\frac{\gamma}{\alpha}$$

$$x^2 + \frac{\beta}{\alpha}x + \left(\frac{\beta}{2\alpha}\right)^2 = -\frac{\gamma}{\alpha} + \left(\frac{\beta}{2\alpha}\right)^2$$

$$\left(x + \frac{\beta}{2\alpha}\right)^2 = -\frac{\gamma}{\alpha} + \left(\frac{\beta}{2\alpha}\right)^2$$

Now, we will solve for x,

$$\left(x + \frac{\beta}{2\alpha}\right)^2 = -\frac{\gamma}{\alpha} + \left(\frac{\beta}{2\alpha}\right)^2$$

$$x + \frac{\beta}{2\alpha} = \pm\sqrt{-\frac{\gamma}{\alpha} + \left(\frac{\beta}{2\alpha}\right)^2}$$

$$x = -\frac{\beta}{2\alpha} \pm \sqrt{-\frac{\gamma}{\alpha} + \left(\frac{\beta}{2\alpha}\right)^2}$$

This is actually solved! Let's just simplify it further,

$$\begin{aligned} x &= \frac{-\beta \pm \sqrt{-\frac{\gamma}{\alpha}(2\alpha)^2 + \left(\frac{\beta}{2\alpha}\right)^2 (2\alpha)^2}}{2\alpha} \\ &= \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} \end{aligned}$$

We are done!
Using quadratic formula for non-trivial equation may involve complicated computations, so we will discuss a more elegant method to solve the quadratics.

Elegant method: The Babylonians
Let $f(x)$ be a quadratic equation of the form,

$$\begin{aligned} \alpha x^2 + \beta x + \gamma &= 0 \\ x^2 + \frac{\beta}{\alpha}x + \frac{\gamma}{\alpha} &= 0 \end{aligned}$$

with $\alpha, \beta, \gamma \in \mathbb{R} \setminus 0$. Then,

$$\begin{aligned} x^2 + \frac{\beta}{\alpha}x + \frac{\gamma}{\alpha} &= (x + x_1)(x + x_2) \\ &= x^2 + (x_1 + x_2)x + x_1 x_2 \end{aligned}$$

We get,

$$\begin{aligned} \text{Sum: } x_1 + x_2 &= \frac{\beta}{\alpha} \\ \text{Product: } x_1 x_2 &= \frac{\gamma}{\alpha} \end{aligned}$$

We will now find the midpoint,

$$\frac{x_1 + x_2}{2} = \frac{\beta}{2\alpha}$$

Now, we know that the constant, $\frac{\beta}{2\alpha}$ is equidistant between the roots x_1 and x_2. Let's denote the equidistant as u, so we get

$$x_1 = \frac{\beta}{2\alpha} - u \tag{0.1}$$

$$x_2 = \frac{\beta}{2\alpha} + u \tag{0.2}$$

and we know that $x_1 x_2 = \frac{\gamma}{\alpha}$, substituting with what we got,

$$\begin{aligned} \left(\frac{\beta}{2\alpha} - u\right)\left(\frac{\beta}{2\alpha} + u\right) &= \frac{\gamma}{\alpha} \\ \left(\frac{\beta}{2\alpha}\right)^2 - u^2 &= \frac{\gamma}{\alpha} \\ u^2 &= \left(\frac{\beta}{2\alpha}\right)^2 - \frac{\gamma}{\alpha} \\ u &= \sqrt{\left(\frac{\beta}{2\alpha}\right)^2 - \frac{\gamma}{\alpha}} \end{aligned}$$

Now, the computation is trivial, by substituting u in (0.1) and (0.2).