Is π transcendental?

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Abstract

In this essay, we will see what transcendental numbers are and provide the required background to prove that π is transcendental. We will use some analytic theories in our proof (trust me it will be more intuitive!).

What is transce... number?

Definition. A transcendental number, α is in $\mathbb{R}$ or $\mathbb{C}$ that is not algebraic,i.e., there does not exist $f(x) \in \mathbb{Q}[x]$ with $\deg(f) = n < \infty$ such that $f(\alpha) = 0$. The best-known transcendental numbers are π and e (which we will show!).

We will review some required background before jumping into the proof.
Let,

$$I(t) = \int_0^t e^{t-u} f(u)\ du$$

where $t \in \mathbb{C}$ and $f(x) \in \mathbb{C}[x]$. Using integration by parts, we get

$$I(t) = e^t \sum_{j=0}^{\infty} f^{(j)}(0) - \sum_{j=0}^{\infty} f^{(j)}(t) = e^t \sum_{j=0}^{n} f^{(j)}(0) - \sum_{j=0}^{n} f^{(j)}(t) \tag{1}$$

where $n = \deg f$. If $f(x) = \sum_{j=0}^{n} a_j x^j$, then we define

$$\bar{f}(x) = \sum_{j=0}^{n} |a_j| x^j$$

Then, we get upper bound,

$$|I(t)| \leq \left| \int_0^t e^{t-u} f(u)\ du \right| \leq |t| \max\{|e^{t-u}|\} \max\{|f(u)|\} \leq |t| e^{|t|} \bar{f}(|t|)$$

Definition (Argument and Symmetric Polynomials). An argument of a function is a value provided to obtain the function's result. A function $f(x_1, x_2)$ of two arguments is a symmetric function if and only if $f(x_1, x_2) = f(x_2, x_1)$, for all x_1 and x_2 such that (x_1, x_2) and (x_2, x_1) are in the domain of f. If the symmetric functions are polynomial functions, then we call them as symmetric polynomials.

Theorem (Fundamental Theorem of Symmetric Functions). Any symmetric polynomial (respectively, symmetric rational function) can be expressed as a polynomial (respectively, rational function) in the elementary symmetric polynomials on those variables.

Lemma 1. If α is an algebraic integer with minimal integer polynomial $f(x) \in \mathbb{Z}[x]$ and if a is the leading coefficient of $f(x)$, then $a\alpha$ is an algebraic integer

Lemma 2. If α is an algebraic integer and $\alpha \in \mathbb{Q}$, then α is an rational integers.

The proof of the fundamental Theorem of Symmetric Functions and other lemmas is left to the reader (it is pretty fun to prove!)

Finally, we will begin our proof.

Theorem. π is a transcendental number.

Proof. We will prove by contradiction. Suppose for the sake of contradiction assume that π is algebraic. Then we can claim that $i\pi$ is also algebraic, since suppose $f(x) \in \mathbb{Z}[x]$ and $f(\pi) = 0$, then $g(x) = f(ix)f(-ix) \in \mathbb{Z}[x]$ and $g(i\pi) = 0$. Therefore, it suffices to prove our following claim,

Main Claim. $\theta = i\pi$ is transcendental.

Suppse θ is algebraic. Let $\deg g(x) = r$, where $g(x)$ is a minimal polynomial for θ and let $\theta_1 = \theta, \theta_2, \ldots, \theta_r$ denote the conjugates of θ. Let b denote denote the leading coefficient of $g(x)$. In particular, $b\theta_j$ is an algebraic integer, by *Lemma 1*.

By Euler formula (most beautiful equation), i.e, $e^{\pi i} = -1$, we deduce that

$$(1 + e^{\theta_1})(1 + e^{\theta_2}) \cdots (1 + e^{\theta_r}) = 0$$

Multiplying the expression on the left out, we get a sum of 2^r term of the form e^{ϕ}, where ϕ is the linear combination of conjugates of θ, i.e., $\phi = e_1\theta_1 + \cdots + e_r\theta_r$ where $e_j \in \{0, 1\}$ for all $j \in \{1, \ldots, r\}$. Let $\phi_1, \ldots, \phi_n$ denote the non-zero expressions of this form so that(since the remaining $2^r - n$ values of ϕ are 0).

$$q + e^{\phi_1} + \cdots + e^{\phi_r} = 0$$

where $q = 2^r - n$. Let p be a large prime, and let

$$f(x) = b^{np} x^{p-1} (x - \phi_1)^p \cdots (x - \phi_n)^p$$

By the fundamental theorem of elementary symmetric functions and Lemma 1 and Lemma 2, $f(x) \in \mathbb{Z}[x]$, to see this more accurately, consider $\phi_1, \ldots, \phi_{2^r}$ as the complete set of ϕ's as above (so the first n are still the non-zero ones) and use that,

$$\prod_{j=1}^{2^r} (x - \phi_j) = x^{2^r - n} \prod_{j=1}^{n} (x - \phi_j)$$

is symmetric in $\theta_1, \ldots, \theta_r$. Define,

$$J = \sum_{i=1}^{n} I(\phi_i)$$

From (1), we deduce that

$$J = -q\sum_{j=0}^{m} f^{(j)}(0) - \sum_{j=0}^{m}\sum_{k=1}^{n} f^{(j)}(\phi_k)$$

where $m = (n+1)p-1$. Observe that the sum over k is a symmetric polynomial in $b\phi_1, \ldots, b\phi_n$ with integer coefficients and thus a symmetric polynomial with integer coefficients in the 2^r-numbers $b\phi = b(e_1\theta_1 + \cdots + e_r\theta_r)$. Hence, by the fundamental theorem of elementary symmetric functions, we obtain that this sum is a rational number. Observe that Lemma 1 and Lemma 2, imply that the sum is further more a rational integer. Since $f^{(j)}(\phi_k) = 0$ for $j < p$, we deduce that the double sum in the expression for J above is a rational integer divisible by $p!$. Observe that $f^{(j)}(0) = 0$ for $j < p-1$ and $p! \mid f^{(j)}(0)$, for $j \geq p$. Also,

$$f^{(p-1)}(0) = b^{np}(-1)^{np}(p-1)!(\phi_1\cdots\phi_n)^p$$

From the fundamental theorem of symmetric functions and Lemma 1 and Lemma 2, we deduce that $f^{(p-1)}(0)$ is a rational integer divisible by $(p-1)!$. Furthermore, if p is sufficiently large, then $f^{(p-1)}(0)$ is not divisible by p. If also, $p > q$, we deduce that

$$|J| \geq (p-1)!$$

On the other hand, using the upper bound we obtained for $|I(t)|$, we have

$$|J| \leq \sum_{k=1}^{n} |\phi_k| e^{|\phi_k|}\bar{f}(|\phi_k|) \leq c_1 c_2^p$$

for some constants c_1 and c_2. We get a contradiction, completing the proof. □