How to prove isomorphism between vector spaces?

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In this essay, I would want to give a computational technique to prove that two abstract vector spaces are isomorphic. Why is isomorphism very important? Well, it provides a deep understanding of the structural properties of these spaces and allow us to establish meaningful connections between seemingly different vector spaces.

But, what does isomorphism even mean? A vector space $\mathbb{V}$ is said to be isomorphic to a vector space $\mathbb{W}$, if there exists a bijective linear mapping, $L : \mathbb{V} \to \mathbb{W}$. L is called an isomorphism from $\mathbb{V}$ to $\mathbb{W}$ and $\mathbb{V}$ is said to be isomorphic to $\mathbb{W}$, denoted as $\mathbb{V} \cong \mathbb{W}$.

Let $\mathbb{V}$ and $\mathbb{W}$ be abstract vector spaces over a field $\mathbb{F}$, our goal is to prove that $\mathbb{V} \cong \mathbb{W}$.

proof idea. To prove the $\mathbb{V} \cong \mathbb{W}$, we need to prove that L is an isomorphism. So, we will stick to the following three steps, which is comparatively trivial. At an overview, we just need to create a bijective linear mapping, $L : \mathbb{V} \to \mathbb{W}$.

Step 1. We define a mapping, $L : \mathbb{V} \to \mathbb{W}$, where the function is, $v \mapsto w$, $v \in \mathbb{V}$ and $w \in \mathbb{W}$. We just need to prove that L is a linear mapping, ie., for all $u, v \in \mathbb{V}$ and $s, t \in \mathbb{F}$, L is linear if and only if $L(su + tv) = sL(u) + tL(v)$. Which in the computational point of view, is trivial.

Step 2. We need to prove that the linear map, L is injective. So to show injectivity, we need to prove that, for all $u, v \in \mathbb{V}$, if $L(u) = L(v)$ then $u = v$.

Step 3. Finally, we need to prove that the linear map, L is surjective. So to show surjectivity, we need to prove that, if for all $w \in \mathbb{W}$, there exists $v \in \mathbb{V}$ such that $L(v) = w$.

Which kinda provides us with the process to prove that L is an isomorphism from $\mathbb{V}$ to $\mathbb{W}$, hence $\mathbb{V} \cong \mathbb{W}$. □

Since, we are on the topic of isomorphsims, I would like to mention some important theorems on this topic.

Lemma 1. Let $L : \mathbb{V} \to \mathbb{W}$ be a linear mapping. L is injective if and only if $\ker(L) = \{0\}$.

Proof. We will prove both the implications.
$(\Longrightarrow)$: Assume that L is injective. If $x \in \ker(L)$, then $L(x) = 0 = L(0)$ which implies that $x = 0$, since L is injective. Hence $\ker(L) = \{0\}$.
$(\Longleftarrow)$: Assume that $\ker(L) = \{0\}$. Let $u, v \in \mathbb{V}$, if $L(u) = L(v)$, then

$$0 = L(u) - L(v) = L(u - v)$$

Hence, $u - v \in \ker(L)$ which implies that $u - v = 0$. Therefore, $u = v$ and so L is injective. □

Theorem 2. Let $\mathbb{V}$ and $\mathbb{W}$ be finite dimensional vector spaces. $\mathbb{V} \cong \mathbb{W}$ if and only if $\dim \mathbb{V} = \dim \mathbb{W}$.

Proof. We will prove both the implications.
($\Longleftarrow$) : Assume that dim $\mathbb{V}$ = dim $\mathbb{W} = n$. Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis for $\mathbb{V}$, and $\mathcal{C} = \{w_1, \ldots, w_n\}$ be a basis for $\mathbb{W}$. We define a mapping, $L : \mathbb{V} \to \mathbb{W}$ which maps the basis vectors in $\mathcal{B}$ to the basis vectors in $\mathcal{C}$. So, we define

$$L(t_1v_1 + \cdots + t_nv_n) = t_1w_1 + \cdots + t_nw_n$$

To prove the that $\mathbb{V} \cong \mathbb{W}$, we first need to prove that L is a linear map. So, let $x = a_1v_1 + \cdots + a_nv_n$, $y = b_1v_1 + \cdots + b_nv_n \in \mathbb{V}$ and $s, t \in \mathbb{R}$. We have,

$$\begin{aligned}
L(sx + ty) &= L\big((sa_1 + tb_1)v_1 + \cdots + (sa_n + tb_n)v_n\big) \\
&= (sa_1 + tb_1)w_1 + \cdots + (sa_n + tb_n)w_n \\
&= s(a_1w_1 + \cdots + a_nw_n) + t(b_1w_1 + \cdots + b_nw_n) \\
&= sL(x) + tL(y)
\end{aligned}$$

Therefore, L is a linear mapping. We will now prove that L is an injective mapping. If $v = \ker(L)$, then

$$\begin{aligned}
0 = L(v) &= L(c_1v_1 + \cdots + c_nv_n) \\
&= c_1L(v_1) + \cdots + c_nL(v_n) \quad [L \text{ is linear}] \\
&= c_1w_1 + \cdots + c_nw_n
\end{aligned}$$

$\mathcal{C}$ is linearly independent so we get $c_1 = \cdots = c_n = 0$. Therefore, $\ker(L) = \{0\}$ and by Lemma 1, L is injective. Finally, we just need to prove that L is a surjective mapping. Since, we know L is injective, we have $\ker(L) = \{0\}$. By rank-nullity theorem, we have rank(L) = dim $\mathbb{V} - 0 = n$. Consequently, range(L) is an n-dimensional subspace of $\mathbb{W}$ which implies that range$(L) = \mathbb{W}$, hence L is surjective. Thus, L is an isomorphism from $\mathbb{V}$ to $\mathbb{W}$ and so $\mathbb{V} \cong \mathbb{W}$.
($\Longrightarrow$) : Assume that $\mathbb{V} \cong \mathbb{W}$. Then, there exists an isomorphism L from $\mathbb{V}$ to $\mathbb{W}$. Since, L is an bijective, we get range$(L) = \mathbb{W}$ and $\ker(L) = \{0\}$. Thus, the Rank-nullity Theorem gives,

$$\dim \mathbb{W} = \dim(\text{range}(L)) = \text{rank}(L) = \dim \mathbb{V} - \text{nullity}(L) = \dim \mathbb{V}$$

Hence proved. □

The proof not only proves the theorem, but it demonstrates a few additional facts.

1. It shows the intuitively obvious fact that if $\mathbb{V} \cong \mathbb{W}$, then $\mathbb{W} \cong \mathbb{V}$. Typically we can just say that $\mathbb{V}$ and $\mathbb{W}$ is isomorphic.

2. It confirms that we can make an isomorphism from $\mathbb{V}$ to $\mathbb{W}$ by mapping basis vectors of $\mathbb{V}$ to basis vectors of $\mathbb{W}$.

3. Finally, observe that once we had proven that L is injective, we could exploit the Rank-Nullity Theorem and that dim $\mathbb{V}$ = dim $\mathbb{W}$ to immediately get that the mapping is surjective. In particular, it shows us how to prove the following theorem.

Theorem 3. If $\mathbb{V}$ and $\mathbb{W}$ are n-dimensional vector spaces and $L : \mathbb{V} \to \mathbb{W}$ is linear, then L is injective if and only if L is surjective.

We saw that we can make an isomorphism by mapping basis vectors to basis vectors. The following theorem shows that this property actually characterizes isomorphisms.

Theorem 4. Let $\mathbb{V}$ and $\mathbb{W}$ be isomorphic vector spaces and let $\{v_1, \ldots, v_n\}$ be a basis for $\mathbb{V}$. A linear mapping $L : \mathbb{V} \to \mathbb{W}$ is an isomorphism if and only if $\{L(v_1), \ldots, L(v_n)\}$ is a basis for $\mathbb{W}$.

Main Theorem. There exists a basis for the vector space $\mathbb{L}$ of all linear operators $L : \mathbb{V} \to \mathbb{V}$ on a n-dimensional real vector space $\mathbb{V}$.

Proof. To prove our main theorem, we will first need to prove the following claims. Let $\mathbb{L} = \{L_1, \ldots, L_n\}$ be a set of linear operators.
Claim 1. There exists a linear bijective map, ie., isomorphism, $T : \mathbb{L} \to M_n(\mathbb{R})$, such that $\mathbb{L} \cong M_n(\mathbb{R})$. We will prove this claim, since the given mapping is from $\mathbb{L}$ to $M_n(\mathbb{R})$, we define its function as

$$L \in \mathbb{L} \mapsto \begin{bmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{nn} \end{bmatrix} \in M_n(\mathbb{R})$$

Where L_1 is a linear mapping from n-dimensional real vector space, $\mathbb{V} \to \mathbb{V}$. We will now show that T is a linear mapping. Let $L_1, L_2 \in \mathbb{L}$, also let $s, t \in \mathbb{R}$. So, by definition

$$\begin{aligned} T(sL_1 + tL_2) &= \left(\begin{bmatrix} sa_{11} + tb_{11} & \cdots & sa_{n1} + tb_{n1} \\ \vdots & \ddots & \vdots \\ sa_{1n} + tb_{1n} & \cdots & sa_{nn} + tb_{nn} \end{bmatrix} \right) \\ &= \left(\begin{bmatrix} sa_{11} & \cdots & sa_{n1} \\ \vdots & \ddots & \vdots \\ sa_{1n} & \cdots & sa_{nn} \end{bmatrix} + \begin{bmatrix} tb_{11} & \cdots & tb_{n1} \\ \vdots & \ddots & \vdots \\ tb_{1n} & \cdots & tb_{nn} \end{bmatrix} \right) \\ &= s \left(\begin{bmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{nn} \end{bmatrix} \right) + t \left(\begin{bmatrix} b_{11} & \cdots & b_{n1} \\ \vdots & \ddots & \vdots \\ b_{1n} & \cdots & b_{nn} \end{bmatrix} \right) \\ &= sT(L_1) + tT(L_2) \end{aligned}$$

Hence, T is a linear mapping. Finally, to prove that $\mathbb{L} \cong M_n(\mathbb{R})$, we just need to show that T is bijective. To show injectivity, we just need to prove that if $T(L_1) = T(L_2)$, then $L_1 = L_2$, for all $L_1, L_2 \in \mathbb{L}$.

$$T(L_1) = T(L_2) \implies \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{n1} \\ \vdots & \ddots & \vdots \\ \alpha_{1n} & \cdots & \alpha_{nn} \end{bmatrix} = \begin{bmatrix} \beta_{11} & \cdots & \beta_{n1} \\ \vdots & \ddots & \vdots \\ \beta_{1n} & \cdots & \beta_{nn} \end{bmatrix}$$

so, we have $\alpha_{ij} = \beta_{ij}$, where $1 \leq i, j \leq n$. Since, both the matrices are equal, the mapping that produces them are also equal, so we get

$$T(L_1) = T(L_2) \implies L_1 = L_2$$

Hence, we proved that T is a injective mapping. Finally, we just have to prove that T is surjective. We will now prove that for all $A \in M_n(\mathbb{R})$, there exists $L \in \mathbb{L}$ such that $T(L) = A$. We will pick, $A = \begin{bmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{nn} \end{bmatrix} \in M_n(\mathbb{R})$. Then, $T(L) = \begin{bmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{nn} \end{bmatrix}$. Therefore, we proved that T is surjective.
Since, we proved that T is a bijective linear mapping, it is a isomorphism from $\mathbb{L}$ to $M_n(\mathbb{R})$. Hence, $\mathbb{L} \cong M_n(\mathbb{R})$.
Claim 2. There exists a basis for $\mathbb{L}$.
Proving this is pretty trivial. By Claim 1, we proved that there exists a bijective linear mapping, T, so by classic definition of bijection, we know that there exists a linear mapping T^{-1} from $M_n(\mathbb{R})$ to $\mathbb{L}$. Now, to prove that T^{-1} is bijective, we use the result we proved earlier. We know that $\mathbb{L} \cong M_n(\mathbb{R})$, so by definition

of isomorphism, we can claim that it is equivalent to $M_n(\mathbb{R}) \cong \mathbb{L}$ which forms an isomorphism, T^{-1} from $M_n(\mathbb{R})$ to $\mathbb{L}$. Now, Let $\{e_1, \ldots, e_n\}$ denote the set of standard basis vectors for $M_n(\mathbb{R})$. Then by Theorem 4, we know that there exists $T^{-1}(e_1), \ldots, T^{-1}(e_n)$ that form a basis for $\mathbb{L}$, which by definition means, that it is linearly independent and spans $\mathbb{L}$.
Hence proved our main theorem as desired. $\square$