Abstract motivation behind Group Theory!

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Unlike, classic texts on abstract algebra that get right into the axioms of groups. I would like to initially develop an initiution on why group theory exists and the concept that motivates its existence. And that concept is called a symmetry (exactly the one in high/middle school math curriculum). Firstly, I will elevate this concept of symmetry, to an abstract sense and later generalize it, which creates an important field in abstract algebra called group theory.

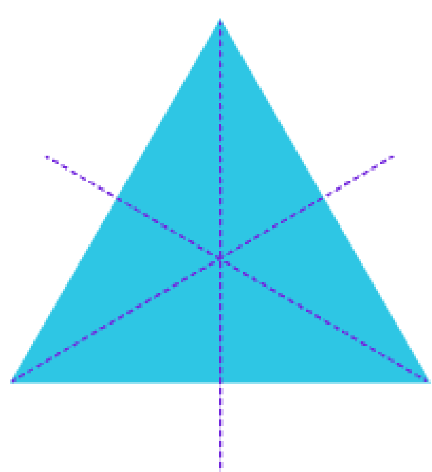
Symmetry

Ideally speaking, group theory is nothing but a mathematical description of symmetry, the structure of the Rubiks cube, simple solutions to otherwise difficult combinatorics problems, the definition of a determinant of a matrix, cryptosystem, distinguishing topological shapes from each other, and much more. Here are some "fancy" jargons on different symmetries.

A *rigid transformation* of an equilateral triangle is a map that maps to the same equilateral triangle. A *identity transformation* is the transformation defined by not changing the position of the triangle vertices at all.

Question 1. *How many rigid transformations does an equilateral triangle (do not count transformations seperately if they are equivalent)?*

Solution. There are three rotations 120°, 240° and 0°. The last of these is the identity transformation, the one that does nothing. We sometimes think of it as a *trivial* rotation. Also there are three rotations, counter-clockwise by 120°, 240° and 0°. These match up with the other three rotations: clockwise 120° which is same as counter-clockwise 240°, and clockwise 240° which is same as counter-clockwise 120°. All that matters is where the points end up, not how they get there. There are also three reflections: the one described above, and the other two corresponding to the other two sides of the triangle.

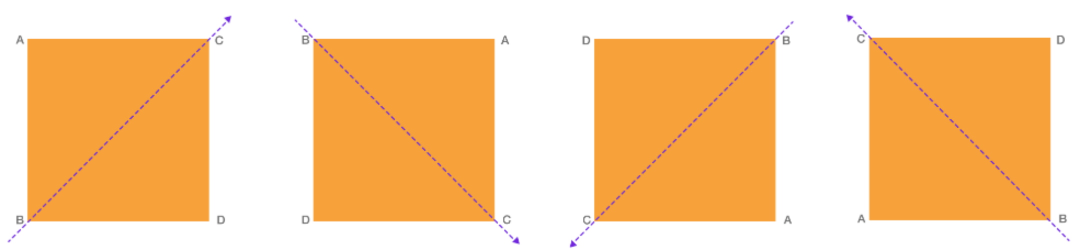


That's **6** rigid transformations of an equilateral triangle. □

We will define a *symmetry* of an object to be a rigid transformation from that object to itself. So, from the last problem, we will say an equilateral triangle has **six** symmetries: three *rotations* and three *reflections*. Since symmetries are rigid transformations, they are really *functions*. We only care about where the points end up, not how they got there. In the example of the triangle, rotating clockwise by $120°$ is considered the same symmetry as rotating clockwise by $360° + 120° = 480°$, and the same symmetry as rotating counter-clockwise by $360° - 120° = 240°$. Also, we will always count the "identity symmetry," corresponding to picking up the triangle, not changing it, and putting it right back down. This corresponds to the identity rigid transformation $f(x) = x$.

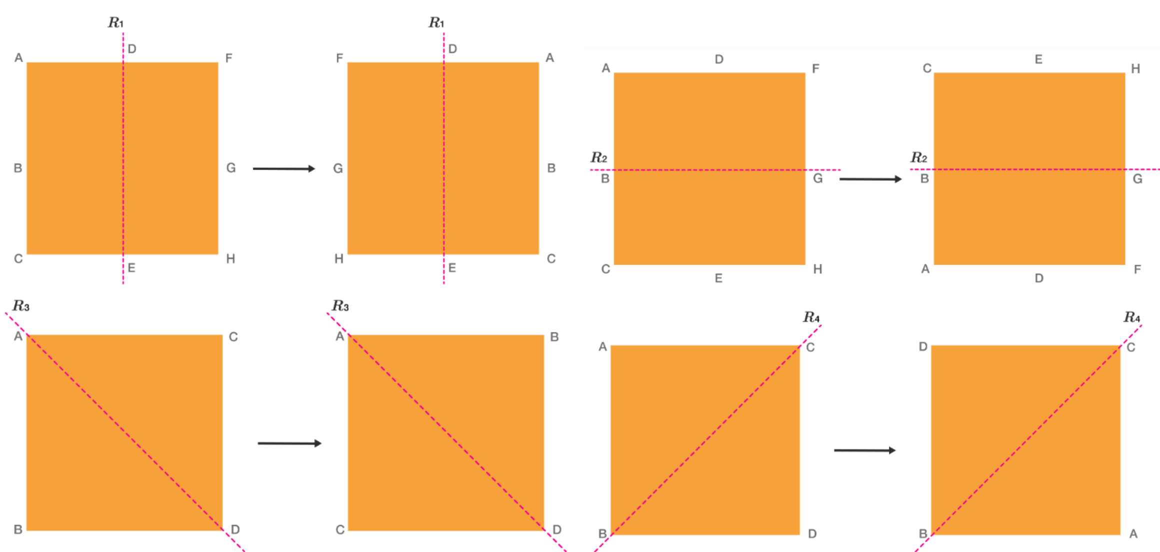
Question 2. *How many rotation and reflection symmetries are there in a square?*

Solution. The symmetries are rotations and reflections. There are four rotations: $90°$, $180°$, $270°$ and I, which is the identity. Below shows on the far left a square with vertices labeled A, B, C, and D as well as the arrow. The second, third and fourth images correspond to clockwise rotations of the square and the arrow by $90°$, $180°$, $270°$ respectively. There are also four reflections. There are two *kinds* of reflections:



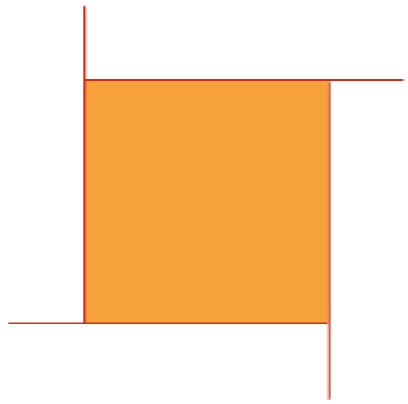
There are horizontal and vertical reflections; these reflect across axes that bisect opposite edges, and there are reflections across axes that connect the corners of the square.

The R in each figure stands for reflection, and the lines represent the axes of symmetry.



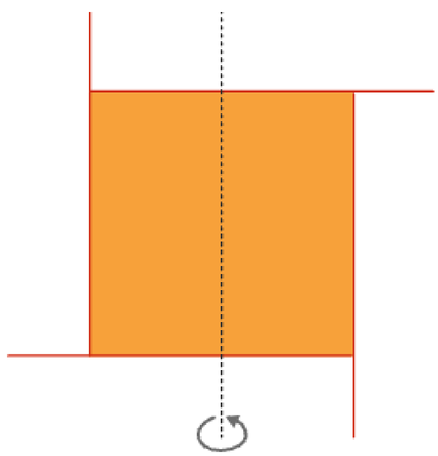
So, there are 4 rotations and 4 reflections, which concludes that there are **8** symmetries. □

Question 3. *Consider a square modified, S', with lines extending past each of its corners in one direction.*

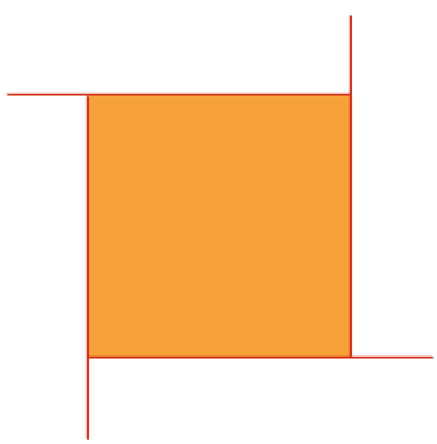


How many symmetries does S' have?

Solution. S' has exactly **4** symmetries, because the four rotations still work, but the four reflections no longer work. The reflections don't work because the external line segments are part of the shape, and reflections don't map this shape to itself. For instance, reflecting across a vertical line maps the original shape.



to



So, we are cleared. □

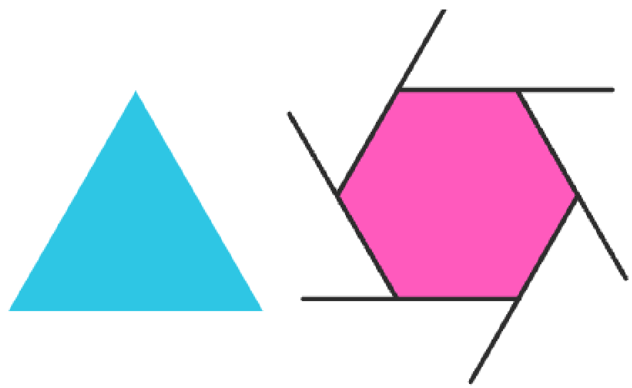
Every shape has the identity transformation for a symmetry. The most asymmetric a shape can be is to have no other symmetries, like the letter R.

Question 4. *How many of the capital letters shown have only the identity symmetry?*

Solution. I will only discuss about F, G, J, K, L, N, P, Q, R, S and Z, since all the other letters are trivial explanations. S, N, and Z are examples where there is 180° rotational symmetry. If you rotate them 180°, you'll get the original letter again. The **non-symmetric letters** are F, G, J, K, L, P, Q, and R. The case of K is somewhat subtle: if the two legs had emerged from the same point, then it would have a vertical reflection symmetry. But the way this K is drawn, with the lower leg emerging from the upper, it does not. □

To recap, every object has a set of symmetries, which are the rigid transformations sending the object to itself. Now, the more symmetries an object has, the more "symmetric" we think it is. But there's more to an object's symmetry than just the number of its symmetries. This is why we can't just use a number to fully describe how symmetric the object is. We will need a more complicated object to measure symmetry: *the group.*

Question 5. *Which of these two objects has more symmetries?*



Proof. Solution Actually, they have the same number of symmetries. We already know that the triangle has 6 symmetries (3 rotations and 3 reflections). There are 6 rotations of the hexagon figure: 60°, 120°, etc. And that's it. If the external line segments hadn't been there, we would also have reflections. (Six of them, in fact.) But, similarly to the modified square from a couple problems ago, because of those external segments, reflections will map this shape into a different shape, so they are not symmetries. □

The two objects from the last problem each have six symmetries specific to their respective shapes (assuming the vertices are labeled); despite their superficial similarities, there's something fundamentally different about the two collections of symmetries.

The difference will involve the idea of doing a transformation multiple times in a row. Notice that if you do a reflection twice in a row, you get back to where you started. We might say that when you do it twice, you get the identity transformation. Similarly, if you rotate the triangle by 120° three times, you get the identity transformation.

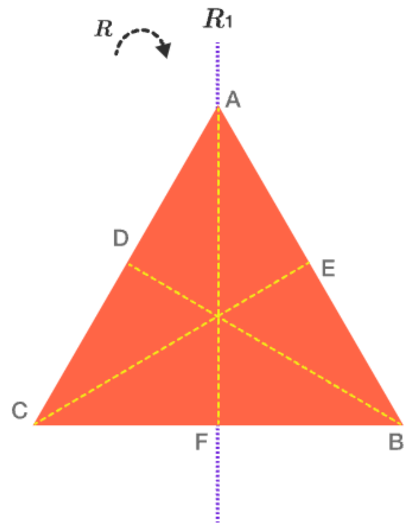
Group theory is an area of algebra, which means that we will be studying how combining objects can make new objects. That's what started to happen in the last problem, when we combined symmetries with themselves.

Remark 0.1. *The hexagon has a symmetry with the property that when you apply it less than six times, you don't get the identity transformation, but when you apply it six times you do, whereas the triangle has no such symmetry. In group theory, only the hexagon has a symmetry S with the property that $S \neq I$, $S^2 \neq I$, $S^3 \neq I$, $S^4 \neq I$ and $S^5 \neq I$, but $S^6 = I$, where I is the identity transformation.*

Combining Symmetries

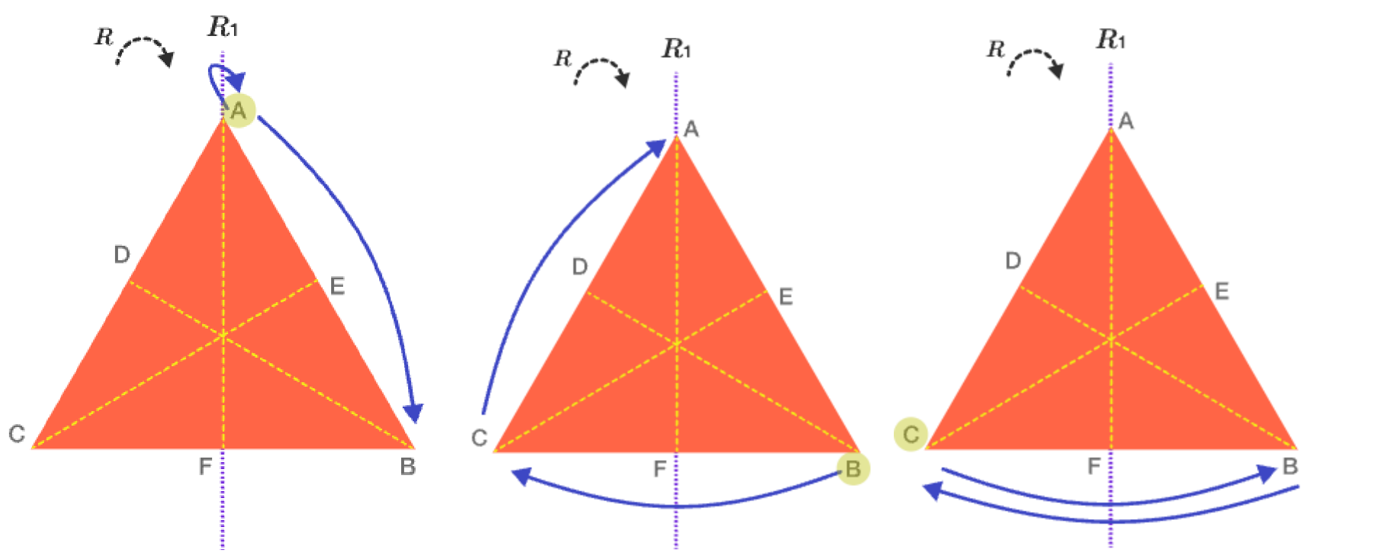
We saw that we could *combine* symmetries by doing one after the other, and thus form a kind of "multiplication" on the symmetries. Now, we'll explore this idea further, and in the next lesson, this will lead to the definition of a group.

Question 6. *Consider the equilateral triangle. Let R be a clockwise rotation by $120°$, and let R_1 be a reflection about the line segment $\overline{AF}$.*



What symmetry do you get when you apply R_1, followed by R?

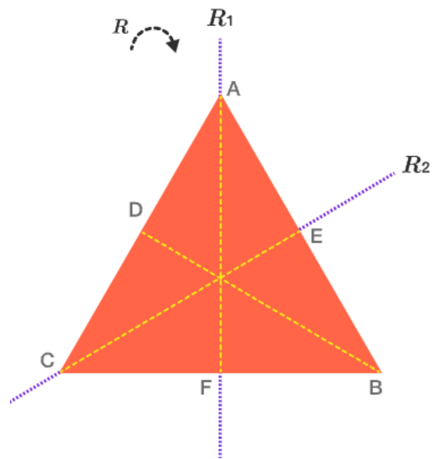
Solution. Let's apply R_1 and then R to each vertex individually.



Vertex A: R_1 sends vertex A to itself, and then R sends A to B. So the whole process sends A to B.
Vertex B: R_1 sends B to C, and then R rotates C to A. So the whole process takes B to A (via the intermediary vertex C).
Vertex C: R_1 sends it to B, and then R rotates B to C. So the whole process takes C to C.
So, it's reflection is about the line segment $\overline{CE}$. □

Now let's combine two reflections. Let R_1 be reflection about the line segment $\overline{AF}$, and let R_2 be reflection about the line segment $\overline{CE}$. Note that after the lines of reflection are defined they do not change position, even when the points of that line segment change position.

Question 7 (Try it yourself!)**.** *What symmetry do you get by applying first R_1 and then R_2?*



In fact, we can combine any two symmetries of a shape by doing one after the other, and obtain a third symmetry of the shape. If A and B are symmetries of a shape, $A * B$, or sometimes just AB or $A \circ B$, denoting the composition symmetry. If we let ϕ_1 denote the reflection about denote the reflection about $\overline{AF}$, let ϕ_2 denote the reflection about $\overline{CE}$, and let R denote clockwise rotation by 120°, then we can write the equation $\phi_2 * \phi_1 = R$. To really understand the symmetries of an object, we should know not only how many symmetries there are, but what the rules are for how they combine with each other. For the equilateral triangle, we could make a 6×6 "multiplication table" that tells you, for any two of the six symmetries of the triangle, what their product is. In the next problems, we will do exactly that for a slightly simpler object. We will work out the symmetry multiplication table for the letter **I**.

Question 8. *How many symmetries, including the identity symmetry, does the letter **I** have?*

Solution. The letter **I** has 4 symmetries; horizontal reflection H, vertical reflection V, rotation by 180°, and the identity transformation I. □

Question 9 (Try it yourself!)**.** *What are $H * H$, $V * V$, and $R * R$?*

So far, our multiplication/group table for the symmetries of the letter **I** looks like this,

	I	**H**	**V**	**R**
I	I	H	V	R
H	H	I	R	V
V	V	R	I	H
R	R	V	H	I

This table completely describes all products of the four symmetries of the letter **I**. Suppose we only have two numbers, 0 and 1, and suppose we consider mod 2 addition on these numbers such that,

$$0 + 0 = 0$$
$$0 + 1 = 1$$
$$1 + 1 = 0$$

The table for this operation is,

	0	**1**
0	0	1
1	1	0

Now, for ordered pairs of 0's and 1's, and mod 2 addition,

$$(0, 1) + (1, 0) = (1, 1)$$
$$(1, 1) + (1, 1) = (0, 0)$$

Group table for odered pairs is,

	(0, 0)	**(1, 0)**	**(0, 1)**	**(1, 1)**
(0, 0)	(0, 0)	(1, 0)	(0, 1)	(1, 1)
(1, 0)	(1, 0)	(0, 0)	(1, 1)	(0, 1)
(0, 1)	(0, 1)	(1, 1)	(0, 0)	(1, 0)
(1, 1)	(1, 1)	(0, 1)	(1, 0)	(0, 0)

Notice the connection between the addition table of ordered pairs and the multiplication table for the letter **I**. To make the connection more evident, we will rename the four ordered pairs, let's denote $i = (0, 0)$, $a = (1, 0)$, $b = (0, 1)$ and $c = (1, 1)$. Then the addition group table becomes,

	i	**a**	**b**	**c**
i	i	a	b	c
a	a	i	c	b
b	b	c	i	a
c	c	b	a	i

This is the same table as the symmetries for the letter **I**, just with different variable names. The *algebraic structure* of the symmetries of the letter **I** is the same as the algebraic structure of ordered pairs of 0's and 1's under mod 2 addition. There is some underlying mathematical object that represents this structure called a *group*. This particular table will be represented by something called the Klein four-group.