Proof of Burnside's Theorem

1 Introduction

In 1904, Burnside proved the following result.

Theorem 1.1. *Let G be a group of size $p^a q^b$, where p and q are primes. Then G is solvable.*

The original proof uses representation theory of finite groups. Here, we prove the above theorem in the case where p and q are odd primes. The proof is purely group-theoretic.

2 Preliminaries

We first set up some necessary notations. All groups to be considered here are finite. Given a subset $X \subseteq G$, we denote the centralizer and normalizer of X (with respect to G) respectively by

$$C_G(X) = \{g \in G : \forall x \in X, gx = xg\},$$

$$N_G(X) = \{g \in G : gXg^{-1} = X\}.$$

Note that we always have $C_G(X) \subseteq N_G(X)$. The center of G is $Z(G) := C_G(G)$.

Given a subgroup H of G, we say that H is a *characteristic subgroup* of G, denoted by $H \operatorname{char} G$, if H is invariant under any automorphism ϕ of G. By invariant, we mean $\phi(H) \subseteq H$; in the case where G is finite, this is the same as $\phi(H) = H$. In particular, since the conjugation action $x \mapsto gxg^{-1}$ is an automorphism for any $g \in G$, any characteristic subgroup of G is normal. An example of a characteristic subgroup of G is $Z(G)$. Here are some properties of characteristic subgroup:

- Given $K \operatorname{char} H \operatorname{char} G$, we have $K \operatorname{char} G$.
- Given $K \operatorname{char} H \lhd G$, we have $K \lhd G$.

Given $x, y \in G$, the *commutator* of x and y is $[x, y] = x^{-1}y^{-1}xy$. Given $X, Y \subseteq G$, we denote by $[X, Y]$ the subgroup generated by $[x, y]$ across all $x \in X$ and $y \in Y$. Define the sequence

$$L_1(G) = G, \quad L_2(G) = [G, G], \quad L_3(G) = [L_2(G), G], \ldots,$$

where $L_{i+1}(G) = [L_i(G), G]$ for all $i \geq 1$. A group G is said to be *nilpotent* if $L_m(G) = 1$ for some $m \geq 1$. Here, we mainly use the following facts about nilpotent groups.

- The direct product of nilpotent groups is nilpotent.
- Every p-group is nilpotent.
- If G is nilpotent and $H \subseteq G$, then $H \subseteq N_G(H)$.
- If H and K are normal nilpotent subgroups of G, then so is HK.

The last fact implies that there is a normal nilpotent subgroup of G that contains all other normal nilpotent subgroups. This subgroup is called the *Fitting subgroup* of G, and is denoted $F(G)$. It is easy to see that $F(G)$ is a characteristic subgroup of G. We take as a fact that:

Lemma 2.1. *If G is finite solvable, then $F(G)$ is non-trivial and $C_G(F(G)) \subseteq F(G)$.*

Given a finite group G and a prime p, we denote by $\mathrm{Syl}_p(G)$ the set of all Sylow p-subgroups of G. Next, for any finite group G and a prime p, we define

$$O_p(G) = \bigcap_{H \in \mathrm{Syl}_p(G)} H.$$

The important property of $O_p(G)$ is that it is the unique maximal normal p-subgroup of G. That is, it is a normal p-subgroup, and every normal p-subgroup of G is contained in $O_p(G)$. Normality follows since any conjugate of a Sylow p-subgroup is another Sylow p-subgroup. Maximality follows from the fact that all Sylow p-subgroups are conjugates of each other.

Lemma 2.2. *A finite group is nilpotent if and only if it is the direct product of its Sylow subgroups.*

Lemma 2.3. *Let G be a solvable group of order $p^a q^b$, where p and q are primes. Let H be a p-subgroup of G. Then $O_q(N_G(H)) \subseteq O_q(G)$.*

Finally, for any prime p and positive integer n, we denote by $\nu_p(n)$ the p-adic valuation of n.

3 Proof of Theorem 1.1 for odd case

We now prove Theorem 1.1 when p and q are odd. We proceed by contradiction.

Let G be a minimal counterexample to Theorem 1.1. Write $|G| = p^a q^b$, where p and q are odd primes and $a, b \geq 0$. It is known that p-groups are solvable for any p prime, so a and b are positive. Here are some properties of G.

- All proper subgroups of G are solvable.

  This follows from minimality of G since the size of a proper subgroup divides $p^a q^b$.

- G is simple.

  Indeed, otherwise we can find a non-trivial normal subgroup $N \lhd G$. Then N and G/N are solvable, and thus G is solvable; a contradiction.

- $O_p(G) = O_q(G) = \{1\}$.

  Indeed, they are normal subgroups of G and are not equal to G.

Here is a less straightforward fact about G. Given subgroups $H, K \leq G$, we say that H normalizes K if $H \subseteq N_G(K)$. That is, $hKh^{-1} = K$ for any $h \in H$.

Lemma 3.1. *Let $P \in \mathrm{Syl}_p(G)$ and Q' be a q-subgroup of G. If P normalizes Q', then $Q' = \{1\}$.*

Proof. Fix $P \in \mathrm{Syl}_p(G)$ and a q-subgroup $Q' \leq G$. Let Q be a Sylow q-subgroup of G containing Q'. Since P normalizes Q', we have $Q' = xQ'x^{-1} \subseteq xQx^{-1}$ for any $x \in P$.

Now, we can check that $PQ = G$ using size constraint. Thus, any $g \in G$ can be written as xy for some $x \in P$ and $y \in Q$. Then $gQg^{-1} = xyQy^{-1}x^{-1} = xQx^{-1}$. This means that every Sylow q-subgroup of G is of form xQx^{-1} for any $x \in P$. The previous paragraph implies that Q' is contained in all Sylow q-subgroups of G. Thus, $Q' \subseteq O_q(G) = \{1\}$, as desired. □

For the next big result, we need a few lemmas.

Lemma 3.2. *Let $P \leq G$ be a p-group. Let $Q \leq N_G(P)$ be a non-cyclic abelian q-group. Then*

$$P = \prod_{x \in Q \setminus \{1\}} C_P(x).$$

Lemma 3.3. *Let $P \leq G$ be an abelian p-group. Let $Q \leq N_G(P)$ be a q-group. Then $P = C_P(Q) \times [P, Q]$.*

Lemma 3.4. *A p-group has a unique subgroup of order p if and only if it is cyclic.*

A *maximal subgroup* of G is a proper subgroup M such that the only subgroups of G containing M are G and M itself. Every proper subgroup of G is contained in a maximal subgroup.

Lemma 3.5. *Let M be a maximal subgroup of G. Then $F(M) = F(M)_p \times F(M)_q$, where $F(M)_p$ and $F(M)_q$ are the unique Sylow p-subgroup and q-subgroup of $F(M)$, respectively. Furthermore, both $F(M)_p$ and $F(M)_q$ are characteristic subgroups of $F(M)$.*

Proof. Recall by definition that $F(M)$ is nilpotent. By Lemma 2.2, we have $F(M) \cong F_p \times F_q$, where F_r is the Sylow r-subgroup of $F(M)$ for $r \in \{p, q\}$. Since F_p and F_q has coprime sizes, they must be a characteristic subgroup of $F(M)$. □

Theorem 3.6. *Let M be a maximal subgroup of G. Then the Fitting subgroup $F(M)$ of M is a p-group or q-group.*

Proof. For convenience, write $F = F(M)$, $F_p = F(M)_p$ and $F_q = F(M)_q$. Lemma 3.5 gives us $F = F_p \times F_q$. We proceed by contradiction, assuming that M is neither a p-group nor a q-group. That is, both F_p and F_q are non-trivial. We prove a series of results and arrive at a contradiction.

First, we prove that M is the unique maximal subgroup of G containing $Z(F)$. In particular, any proper subgroup of G containing $Z(F)$ is contained in M. Let M' be an arbitrary maximal subgroup of G containing $Z(F)$. Notice that since $F = F_p \times F_q$, we have

$$Z(F_p) \subseteq Z(F) \subseteq F \subseteq M_1.$$

We also have $Z(F_p) \subseteq C_G(Z(F_q))$, using the fact that $A \subseteq C_{A \times B}(B)$ for any groups A and B. Thus we get $Z(F_p) \subseteq C_{M_1}(Z(F_q)) \subseteq N_{M_1}(Z(F_q))$.

Now notice that $Z(F_p) \operatorname{char} F_p \operatorname{char} F \operatorname{char} M$, so $Z(F_p) \lhd M$. Similarly, $Z(F_q) \lhd M$, so M normalizes $Z(F_q)$. Since G is simple and $Z(F_q) < G$, we get $N_G(Z(F_q)) = M$. In particular, $N_{M_1}(Z(F_q)) \leq M$. Combining with $Z(F_p) \subseteq N_{M_1}(Z(F_q))$ yields $Z(F_p) \lhd N_{M_1}(Z(F_q))$. Since $Z(F_p)$ is a p-group, we get $Z(F_p) \subseteq O_p(N_{M_1}(Z(F_q)))$. Lemma 2.3 then yields $Z(F_p) \subseteq O_p(M_1)$, and similarly $Z(F_q) \subseteq O_q(M_1)$.

By Lemma 3.5, we can write $F(M_1) = F(M_1)_p \times F(M_1)_q$. Recalling that $O_p(M_1)$ is normal nilpotent, we have

$$Z(F_p) \subseteq O_p(M_1) \subseteq F(M_1)_p \subseteq C_{M_1}(F(M_1)_q) \subseteq C_G(F(M_1)_q).$$

This implies

$$F(M_1)_q \subseteq C_G(Z(F_p)) \subseteq N_G(Z(F_p)) = M.$$

Similarly $F(M_1)_p \subseteq M$, so $F(M_1) \subseteq M$. In particular, $Z(F(M_1)) \subseteq M$. We can repeat the same argument to get $F = F \subseteq M_1$.

Now one sees that $F_p \lhd N_G(F_q)$. Since $F_p \subseteq M_1$, this yields $F_p \lhd N_{M_1}(F_q)$. Thus $F_p \subseteq O_p(N_{M_1}(F_q))$, and Lemma 2.3 yields $F_p \subseteq O_p(M_1)$. Then F is a normal nilpotent subgroup of M_1, which implies $F \subseteq F(M_1)$. Similarly, we also get $F(M_1) \subseteq F$, so $F = F(M_1)$. Thus we get $M_1 = N_G(F(M_1)) = N_G(F) = M$, as desired.

Next, we show that F is not cyclic. Here we use the assumption that F_p and F_q are non-trivial. Suppose for the sake of contradiction that F is cyclic. Then for $r \in \{p, q\}$, F_r is cyclic. Thus it contains a unique subgroup of order r, say U_r, by Lemma 3.4. For this proof, WLOG let $p < q$.

Since $U_p \operatorname{char} M$, U_pQ is a subgroup of M and so $Q \in \mathrm{Syl}_q(U_pQ)$. Let n_q be the number of conjugates of Q over U_pQ. By Sylow's theorem, $n_q \equiv 1 \pmod{q}$ and $n_q \mid [U_pQ : Q] = p$. But $p < q$, so this forces $n_q = 1$; that is, $Q \lhd U_pQ$. In particular, U_p normalizes Q, so $[Q, U_p] \subseteq U_p \cap Q$, which is trivial since $|U_p| = p$ and $|Q|$ are coprime. That is, Q and U_p commutes; in particular, $U_pQ = U_p \times Q$ and $U_p \subseteq C_{F_p}(Q)$.

Since $F_p \operatorname{char} F \operatorname{char} M$, we have $F_p \lhd M$, so $N_M(F_p) = M$. Then $Q \subseteq N_M(F_p)$ and F_p is abelian, so Lemma 3.3 yields $F_p = C_{F_p}(Q) \times [F_p, Q]$. But F_p is cyclic and both factors $C_{F_p}(Q)$ and $[F_p, Q]$ are p-groups. Since $U_p \subseteq C_{F_p}(Q)$, this forces $[F_p, Q] = \{1\}$. So Q and F_p commutes, i.e. $Q \subseteq C_G(F_p)$. Then $Z(Q) \subseteq C_M(F) \subseteq F$, and thus $\{1\} \neq Z(Q) \subseteq F_q$. This implies $U_q \operatorname{char} Z(Q) \operatorname{char} Q$.

Now pick some $Q' \in \mathrm{Syl}_q(G)$ containing Q. Then $Q \subsetneq N_{Q'}(Q)$. Choose an arbitrary $P \in \mathrm{Syl}_p(M)$. Then $|P||N_{Q'}(Q)| > |P||Q| = |M|$, and so $\langle P, N_{Q'}(Q)\rangle \supsetneq M$. By maximality, we have $\langle P, N_{Q'}(Q)\rangle = G$. On the other hand, $\{1\} \neq F_q \lhd \langle P, N_{Q'}(Q)\rangle = G$. A contradiction, since F_q is a proper subgroup of G.

We are now at the final step. Suppose for the sake of contradiction that F_p and F_q are both non-trivial. Since F is not cyclic, F_p does not contain a unique subgroup of order p. That means F_p contains a non-cyclic subgroup V isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, where one of the component is $Z(F_p)$.

For each $x \in V \setminus \{1\}$, we have $Z(F) \subseteq C_G(x) \neq G$. Thus, maximality of M yields $C_G(x) \subseteq M$ for all $x \in V \setminus \{1\}$. Fix some $P \in \mathrm{Syl}_p(M)$ containing V.

Let Q_0 be any q-subgroup of G normalized by P. By Lemma 3.2, we have

$$Q_0 = \prod_{x \in V \setminus \{1\}} C_{Q_0}(x) \subseteq M.$$

Now let Q be a Sylow q-subgroup of M containing Q_0. Let $Q_0^M = \bigcup_{x \in M} x^{-1} Q_0 x$. Since a Sylow p-subgroup normalizes Q_0 (namely P), we have

$$Q_0^M = \bigcup_{x \in Q} x^{-1} Q_0 x \subseteq Q.$$

Clearly, $Q_0^M \lhd M$, so we get that Q_0^M is a normal nilpotent q-subgroup of M. Thus, $Q_0 \subseteq Q_0^M \subseteq F_q$. That is, any q-subgroup of G normalized by P is contained in F_q. Note that F_q is normalized by P, so this means F_q is the (unique) maximal q-subgroup of G normalized by P. Since F_q is non-trivial by assumption, Lemma 3.1 yields $P \notin \mathrm{Syl}_p(G)$.

Now pick an arbitrary $P' \in \mathrm{Syl}_p(G)$ containing P. Then $P \subsetneq N_{P'}(P)$, but also $N_{P'}(P)$ is a p-subgroup. Then $N_{P'}(P) \nsubseteq M$ and thus $N_G(P) \nsubseteq M = N_G(F_q)$. On the other hand, for any $x \in N_G(P)$, we have

$$P = x^{-1} P x \subseteq x^{-1} M x = x^{-1} N_G(F_q) x = N_G(x^{-1} F_q x).$$

Since $x \in N_G(P)$, one can check that $x^{-1} F_q x$ is also normalized by P. Then maximality of F_q yields $x^{-1} F_q x \subseteq F_q$, and thus equality occurs. That is, $x \in N_G(F_q) = M$; a contradiction. □

For the next big result, for any p-group P, let $J(P)$ the subgroup of P generated by all abelian subgroups of P with maximal order. Clearly, any automorphism of P maps an abelian subgroup to another abelian subgroup of the same order. From this, one can see that $J(P) \operatorname{char} P$. Also, note that $J(P)$ is a non-trivial p-group, so $Z(J(P))$ is non-trivial as well.

Lemma 3.7. *Let M be a solvable group of order $p^a q^b$. Suppose that $O_p(M) \neq \{1\}$ but $O_q(M) = \{1\}$. Then, for any $P \in \mathrm{Syl}_p(M)$, we have $Z(J(P)) \lhd M$. That is, M normalizes $Z(J(P))$.*

Corollary 3.8. *Let M be a maximal subgroup of G. By Theorem 3.6, $F(M)$ is an r-group for some $r \in \{p, q\}$. Then for any $P \in \mathrm{Syl}_r(M)$, we have $N_G(Z(J(P))) = M$. That is, M is exactly the normalizer of $Z(J(P))$.*

Proof. WLOG let $r = p$. Since $O_q(M)$ normal nilpotent, we have $O_q(M) \subseteq F(M)$. But $O_q(M)$ is a q-group and $F(M)$ is a p-group, so $O_q(M) = \{1\}$. On the other hand, by Lemma 2.1, $F(M)$ is non-trivial normal, so $O_p(M)$ is non-trivial. Lemma 3.7 yields $M \subseteq N_G(Z(J(P)))$. But $Z(J(P))$ is non-trivial and $Z(J(P)) \leq P < G$. Since G is simple, maximality of M yields $N_G(Z(J(P))) = M$. □

Lemma 3.9. *Let M be a maximal subgroup of G. By Theorem 3.6, $F(M)$ is an r-group for some $r \in \{p, q\}$. Then M contains a Sylow r-subgroup of G.*

Proof. Fix a maximal subgroup $M \leq G$, and WLOG let $F(M)$ be a p-group. By Corollary 3.8, M is exactly the normalizer of $Z(J(P))$.

Now pick a Sylow p-subgroup P' of G containing P. Then we have

$$Z(J(P)) \operatorname{char} J(P) \operatorname{char} P \lhd N_{P'}(P) \implies Z(J(P)) \lhd N_{P'}(P).$$

Thus $P \subseteq N_{P'}(P) \subseteq P' \cap M$. But $P' \cap M$ is a p-subgroup and P is a Sylow p-subgroup (of M). Thus, the above subset relation is an equality; $P = N_{P'}(P) = P' \cap M$. Since $P \subseteq P'$ are p-groups, this forces $P = P'$. Thus P is a Sylow p-subgroup of G. □

Lemma 3.10. *Fix an arbitrary $P \in \mathrm{Syl}_p(G)$. Then there is exactly one maximal subgroup of G containing $Z(P)$. In particular, there is also exactly one maximal subgroup of G containing P. (The same statement holds for $Q \in \mathrm{Syl}_q(G)$.)*

Proof. Let M be a maximal subgroup of G containing P. We first prove that it is the only one containing P, and then prove that it is also the only one containing $Z(P)$.

First note that P normalizes $O_q(M)$ since $P \subseteq M$. Lemma 3.1 yields $O_q(M) = \{1\}$, which forces $F(M)$ to be a p-group. Then Corollary 3.8 yields $M = N_G(Z(J(P)))$. That is, $M := N_G(Z(J(P)))$ is the unique maximal subgroup of G containing P.

Now we show that M is the only maximal subgroup of G containing $Z(P)$. Suppose for the sake of contradiction that $Z(P)$ is contained in a maximal subgroup $M' \neq M$. Choose M' that maximizes $\nu_p(|M' \cap M|)$. Let P' be a Sylow p-subgroup of $M' \cap M$ containing $Z(P)$. By the previous paragraph, we know that $P' \notin \mathrm{Syl}_p(G)$. We can choose some $x \in M$ such that $x^{-1}P'x \subseteq P$. The containment is strict since $P' \notin \mathrm{Syl}_p(G)$. Thus we get $x^{-1}P'x \subsetneq N_P(x^{-1}P'x)$.

We first claim that $N_G(P') \subseteq M'$. Otherwise, choose a maximal sub group $M_0 \neq M'$ containing $N_G(P')$. Then $Z(P) \subseteq P' \subseteq M_0$ and thus

$$\nu_p(|M_0 \cap M|) \geq \nu_p(|N_G(P') \cap M|) \geq \nu_p(|N_P(P')|) > \nu_p(|P'|) = \nu_p(|M' \cap M|),$$

where the last equality holds since $P' \in \mathrm{Syl}_p(M' \cap M)$. A contradiction to the choice of M'.

Next, we claim that $x^{-1}P'x \in \mathrm{Syl}_p(M')$. Let P_0 be a Sylow p-subgroup of M' containing $x^{-1}P'x$. If $x^{-1}P'x \neq P_0$, then

$$x^{-1}P'x \subsetneq N_{P_0}(x^{-1}P'x) \subseteq P_0 \cap M \subseteq M' \cap M,$$

and also $N_{P_0}(x^{-1}P'x)$ is a p-group. Then $x^{-1}P'x \notin \mathrm{Syl}_p(M' \cap M)$, but $P' \in \mathrm{Syl}_p(M' \cap M)$; a contradiction.

By Lemma 3.9, since M contains a Sylow p-subgroup (namely P), $F(M)$ is a p-group. Recall that $x^{-1}P'x \subsetneq P$ and $P' \subseteq \mathrm{Syl}_p(M)$. Then the Sylow p-subgroups of M' are not Sylow p-subgroups of G. Thus, $F(M')$ is a q-group and $\mathrm{Syl}_q(M') \subseteq \mathrm{Syl}_q(G)$. By the first step, M' contains a Sylow q-subgroup of G. This implies $G = PM'$. We assumed that $Z(P) \subseteq M'$, and so

$$Z(P) = \bigcap_{x \in P} x^{-1}Z(P)x \subseteq \bigcap_{x \in P} x^{-1}M_1x \subseteq \bigcap_{x \in G} x^{-1}M_1x \lhd G.$$

But the last intersection is smaller than G and $Z(P)$ is non-trivial. This contradicts the fact that G is simple. □

Finally, we prove the following lemma.

Lemma 3.11. *There exists $P_1, P_2 \in \mathrm{Syl}_p(G)$ such that $P_1 \cap P_2 = \{1\}$. (Similarly, there exists $Q_1, Q_2 \in \mathrm{Syl}_q(G)$ such that $Q_1 \cap Q_2 = \{1\}$.)*

Proof. We first there is a Sylow p-subgroup of G not contained in M. Indeed, they are all conjugates of P_1, i.e. of form xP_1x^{-1} for some $x \in G$. Note that xMx^{-1} is a maximal subgroup of G, and it contains xP_1x^{-1}. So, if $xP_1x^{-1} \subseteq M$ for all $x \in G$, then uniqueness in Lemma 3.10 imply $xMx^{-1} = M$ for all $x \in G$. That is, $M \lhd G$; a contradiction, since G is simple.

Now fix some $P_2 \in \mathrm{Syl}_p(G)$ such that $P_2 \nsubseteq M$. Suppose for the sake of contradiction that $P_2 \cap M$ is non-trivial. Note that $P_2 \cap M$ is a p-group contained in M. Thus, there exists $x \in M$ such that

$$x^{-1}(P_2 \cap M)x \subseteq P_1,$$

and so we get

$$Z(P_1) \subseteq N_G(x^{-1}(P_2 \cap M)x) = x^{-1}N_G(P_2 \cap M)x.$$

Since G is simple and $P_2 \cap M$ is a non-trivial subgroup, it is not normal. Thus $N_G(P_2 \cap M) \neq G$, and the above relation and uniqueness of M from Lemma 3.10 yields

$$x^{-1}N_G(P_2 \cap M)x \subseteq M.$$

Since $x \in M$, this implies $N_G(P_2 \cap M) \subseteq M$. Also, it is clear that $P_2 \cap M \subseteq N_G(P_2 \cap M)$, so we get

$$N_{P_2}(P_2 \cap M) = P_2 \cap N_G(P_2 \cap M) = P_2 \cap M.$$

But $P_2 \cap M$ is a proper subgroup of the p-group P_2. This equality cannot hold; contradiction. □

Now we can obtain contradiction using Lemma 3.11. Without loss of generality, let $p^a > q^b$. Then there exists $P_1, P_2 \in \mathrm{Syl}_p(G)$ such that $P_1 \cap P_2 = \{1\}$. But then we get

$$p^{2a} > |G| \geq |P_1P_2| = \frac{|P_1||P_2|}{|P_1 \cap P_2|} = \frac{p^{2a}}{1} = p^{2a},$$

a contradiction! This concludes the proof of Theorem 1.1.