Something that fascinated me!

Sachin Kumar
University of Waterloo, Faculty of Mathematics

I read about this amazing and extremelyyyyyy elegant idea of Srinivasa Ramanujan, when I was in High-school. But, after studying mathematics at university, it was more than just a fun fact but has lot of mathematical implications. In this essay, I will discuss the idea of Ramanujan summation and its connection to Riemann zeta function, $\zeta(s)$.

For those of you who are unfamiliar with this series, which has come to be known as the Ramanujan Summation after a famous Indian mathematician named Srinivasa Ramanujan, it states that if you add all the natural numbers, that is 1, 2, 3, ..., all the way to infinity, you will find that it is equal to $-\frac{1}{12}$. This may look impossible, but don't worry as a mathematician, I will provide a PROOF.

Before, I provide a proof of Ramanujan summation, I give a brief idea of what the Riemann zeta function is,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for $\Re(s) > 1$. This function is holomorphic (or has an analytic continuation) on the complex plane. We need to consider a special of case of the Riemann zeta function at $s = -1$, for the required summation.

$$\zeta(-1) = 1 + \frac{1}{2^{-1}} + \frac{1}{3^{-1}} + \frac{1}{4^{-1}} + \cdots = 1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}$$

I think the connection is made pretty clear, advanced theories in mathematics can be developed with such fascinating facts. Before, I prove the summation, I will justify two equally fascinating facts, that I will prove.

Claim. The following two summation holds,

$$1 - 1 + 1 - 1 + 1 \cdots = \frac{1}{2} \tag{0.1}$$

$$1 - 2 + 3 - 4 + 5 \cdots = \frac{1}{4} \tag{0.2}$$

Proof. Let $A = 1 - 1 + 1 - 1 + 1 \cdots$, then

$$\begin{aligned} 1 - A &= 1 - (1 - 1 + 1 - 1 + 1 \cdots) \\ &= 1 - 1 + 1 - 1 + 1 \cdots \\ &= A \end{aligned}$$

Therefore, $1 - A = A \implies 2A = 1 \implies A = \frac{1}{2}$. We will now prove (0.2). Let $B = 1 - 2 + 3 - 4 + 5 \cdots$, then

$$
\begin{aligned}
A - B &= (1 - 1 + 1 - 1 + 1 \cdots) - (1 - 2 + 3 - 4 + 5 \cdots) \\
&= (1 - 1 + 1 - 1 + 1 \cdots) - 1 + 2 - 3 + 4 - 5 \cdots \\
&= (1 - 1) + (-1 + 2) + (1 - 3) + (-1 + 4) + (1 - 5) \cdots \\
&= 0 + 1 - 2 + 3 - 4 + 5 \cdots \\
&= B
\end{aligned}
$$

Therefore, $A - B = B \implies A = 2B \implies \frac{1}{2} = 2B \implies \frac{1}{4} = B$. We are done! $\square$

These equations does not have a fancy name, since it has proven by many mathematicians over the years while simultaneously being labeled a paradoxical equation. Nevertheless, it sparked a debate amongst academics at the time, and even helped extend Euler's research in the Basel Problem and lead towards important mathematical functions like the Riemann Zeta function. Now, we can prove the main theorem.

Proof. Let $C = 1 + 2 + 3 + 4 + 5 \cdots$, then

$$
\begin{aligned}
B - C &= (1 - 2 + 3 - 4 + 5 \cdots) - (1 + 2 + 3 + 4 + 5 \cdots) \\
&= (1 - 2 + 3 - 4 + 5 \cdots) - 1 - 2 - 3 - 4 - 5 \cdots \\
&= (1 - 1) + (-2 - 2) + (3 - 3) + (-4 - 4) + (5 - 5) \cdots \\
&= 0 - 4 + 0 - 8 + 0 \cdots \\
&= -4 - 8 - 12 \cdots \\
&= -4(1 + 2 + 3 \cdots) \\
&= -4C
\end{aligned}
$$

Therefore, $B - C = -4C \implies B = -3C \implies \frac{1}{4} = -3C \implies C = -\frac{1}{12}$. We are done! $\square$

Now, why this is important. Well for starters, it is used in string theory. Not the Stephen Hawking version unfortunately, but actually in the original version of string theory (called Bosonic String Theory). Now unfortunately Bosonic string theory has been somewhat outmoded by the current area of interest, called supersymmetric string theory, but the original theory still has its uses in understanding superstrings, which are integral parts of the aforementioned updated string theory.

The Ramanujan Summation also has had a big impact in the area of general physics, specifically in the solution to the phenomenon know as the Casimir Effect. Hendrik Casimir predicted that given two uncharged conductive plates placed in a vacuum, there exists an attractive force between these plates due to the presence of virtual particles bred by quantum fluctuations. In Casimir's solution, he uses the very sum we just proved to model the amount of energy between the plates. And there is the reason why this value is so important.

So there you have it, the Ramanujan summation, that was discovered in the early 1900's, which is still making an impact almost 100 years on in many different branches of physics, and can still win a bet against people who are none the wiser.