The legendary analytic function!

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It is well known that, for $\Re(s) > 0$, the series

$$1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

is convergent if and only if $s > 1$. If we allow $s = \sigma + ir$ to be complex, then

$$|n^s| = |n^\sigma||n^{ir}| = |n^\sigma||e^{ir \log n}| = |n^\sigma|$$

and so the series is absolutely convergent if $\sigma > 1$. We define Riemann's zeta function, ζ by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \tag{0.1}$$

for all $\Re(s) > 1$. An immediate connection with number theory is revealed by the following theorem due to Euler, in which P denotes the set of primes.

Theorem. For all $\Re(s) > 1$,

$$\frac{1}{\zeta(s)} = \prod_{p \in P} \left(1 - \frac{1}{p^s}\right)$$

Proof. Observe first that,

$$\zeta(s)\left(1 - \frac{1}{2^s}\right) = \left(1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots\right)\left(1 - \frac{1}{2^s}\right) = 1 + \frac{1}{3^s} + \frac{1}{5^s}$$

all terms, $\frac{1}{n^s}$, where n is even, is being omitted. Next,

$$\zeta(s)\left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right) = 1 + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \cdots$$

where, now we are leaving out $\frac{1}{n^s}$ for all multiplies of 2 and 3. If p_k is the k-th prime, we see that

$$\zeta(s)\left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\cdots\left(1 - \frac{1}{p_k^s}\right) = 1 + \sum_{n \in D_k} \frac{1}{n^s}$$

where $D_k = \{x \in \mathbb{N} : 2, 3, \ldots, p_k \nmid x\}$. Hence,

$$\left|\zeta(s)\left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\cdots\left(1 - \frac{1}{p_k^s}\right) - 1\right| \leq \left|\frac{1}{(p_k+1)^s}\right| + \left|\frac{1}{(p_k+2)^s}\right| + \cdots$$

and this tends to 0 as $k \to \infty$. Hence,

$$\zeta(s) = \prod_{p \in P}\left(1 - \frac{1}{p^s}\right) = 1$$

as required. $\square$

If you have already come across the gamma function, then

$$\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}\, dx \tag{0.2}$$

for all $s > 0$, and here too we can allow s to be complex and regard function as defined whenever $\Re(s) > 0$. It is trivial to prove that,

$$\Gamma(s) = (s-1)\Gamma(s-1) \tag{0.3}$$

for all $\Re(s) > 1$, and we can use this functional equation backwards to defines $\Gamma(s)$ for $\Re(s) < 0$: if $\Re(s+n) \in (0,1)$, then

$$\Gamma(s) = \frac{\Gamma(s+n)}{s(s+1)\cdots(s+n-1)} \tag{0.4}$$

This fails if $s \leq 0$, and in fact it can be proven that Γ is meromorphic function with simples poles at 0, -1, -2, ... Substituting, $x = nu$ in the integral (0.2) gives,

$$n^{-s}\Gamma(s) = \int_0^\infty e^{-nu}u^{s-1}du$$

and summing from 1 to ∞ gives,

$$\begin{aligned}\zeta(s)\Gamma(s) &= \sum_{n=1}^{\infty}\left[\int_0^\infty e^{-nu}u^{s-1}\, du\right] = \int_0^\infty (e^{-u} + e^{-2u} + \cdots)u^{s-1}\, du \\ &= \int_0^\infty \frac{e^{-u}u^{s-1}\, du}{1-e^{-u}}\end{aligned}$$

It follows that, for all $\Re(s) > 1$

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{u^{s-1}}{e^u - 1}\, du$$

We can kinda make it more complicated,

$$\zeta(s) = \frac{i\Gamma(1-s)}{2\pi}\int_C \frac{(-z)^{s-1}}{e^z - 1}\, dz \tag{0.5}$$

where C is the limiting contour beginning and ending at $+\infty$ on the x-axis, encircling the origin once in a positive direction, but slender enough to exclude the poles $\pm 2i\pi$, $\pm 4i\pi$, ... of the integrand. We interpret $(-z)^{s-1}$ in the usual way as $e^{s-1}\log(-z)$, noting that the cut for $\log(-z)$ lies along the positive x-axis. The formula (0.5) makes sense for all $s \in \mathbb{C}$, except possibly for the poles 2, 3, 4, ... of $\Gamma(1-s)$, but we already know that $\zeta(s)$ is defined at these points. In fact, we now have $\zeta(s)$ defined as a meromorphic function over the whole of $\mathbb{C}$, with a single simple pole at $s = 1$. By developing these ideas a little further, one obtains a functional equations for ζ, somewhat more complicated than (0.3) for the gamma-function:

$$\zeta(s) = 2^{s-1}[\Gamma(s)]^{-1}\sec\left(\frac{\pi s}{2}\right)\zeta(1-s)$$

At each negative integer $[\Gamma(s)]^{-1}$ has a zero of order 1. If the integer is odd, then this is cancelled by the pole of order 1 for $\sec(\frac{\pi s}{2})$, but if $s = -2, -4, \ldots$ we have $\zeta(s) = 0$. In fact those are the only zeroes of ζ in the region $\{s : \Re(s) < 0\}$. From (0.1) it is not hard to deduce that there are no zeroes of ζ in the region $\{s : \Re(s) < 0\}$, and so we have the conclusion that the remaining zeroes of ζ lie in the strip

$\{s : 0 \le \Re(s) \le 1\}$ lie on the line $\{s : \Re(s) = \frac{1}{2}\}$, and this became known as our famous millenium prize conjecture, the **Riemann Hypothesis**. It is something of a puzzle that this is still unproved, for complex analysis is replete with powerful results and techniques. The late 20th saw the proof of several of the classical unsolved problems, notably the Four Colour Theorem and the Fermat Last Theorem, but the Riemann Hypothesis has so far resisted all attempts. As early as 1914 Hardy proved that ζ has infinitely many zeroes on the line $\Re(s) = \frac{1}{2}$, and nobody seriously believes that Riemann's guess is incorrect. A much weaker version of the Hypothesis is that there is no zeroes of ζ on the line $\Re(s) = 1$, and it was by proving this result that Hadamard and de la Vallée Poussin we able to establish the Prime Number Theorem: if $\pi(x)$ is defined as $|\{p \in P : p \le x\}|$, then

$$\pi(x) \sim \frac{x}{\log x}$$

A precise error term in this formula would follow from the full Riemann Hypothesis. There is an extensive literature on consequences of the Riemann Hypothesis, which is not silly as it might see at first sight. Titchmarsh, in his book ”The zeta-function of Riemann”, at the beginning of the final ”consequences” chapter, puts the case very well:

”If the Riemann Hypothesis is true, it will presumably be proved some day. These theorems will then take their place as an essential part of the theory. If it is false, we may perhaps hope in this way sooner or later to arrive at a contradiction. Actually the theory, as far as it goes, is perfectly coherent, and shews no sign of breaking down.”

As the spelling ”shews” might suggest, Titchmarsh was writing in 1930, but his summary is just as true in 2003.