IMO 1988 Problem 6

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Abstract

No words can explain its beauty, elegance and non-triviality!

Question. Let $a, b \in \mathbb{N}$, with $b \geq a$. Assume that

$$k = \frac{a^2 + b^2}{ab + 1}$$

is an integer. Prove that k is a square. The proof method here is reduction (or the Fermat's method of descent).

Proof. We will prove our statement using the technique (proof method) of reduction (or Fermat's method of descent), which is a subset of a well-known technique named Viéta Jumping (or root jumping).
Let assume that $(a', b') \in \mathbb{N}^2$, with $b' \geq a'$ be a minimal postive integer solution to

$$k = \frac{x^2 + y^2}{xy + 1}, \quad x, y \in \mathbb{N} \tag{1}$$

here the minimal is among all values of $x + y$. Let there exist $b^* \in \mathbb{N}$, such that $b^* = ka' - b'$. So, We will claim the following,

Claim 1. *(a', b^*) is a solution to*

$$k = \frac{x^2 + y^2}{xy + 1}, \quad x, y \in \mathbb{N}$$

and $0 < b^* < b'$.

We will first prove that $b^* < b'$. By assumption, we know that $a' \leq b'$ and we are given that $b^* = ka' - b'$. So, by squaring both sides we get

$$\begin{aligned}
(a')^2 &\leq (b')^2 \\
(a')^2 + (b')^2 &\leq 2(b')^2 \\
(a')^3 + (a')(b')^2 &\leq 2(a')(b')^2 \\
(a')^3 + (a')(b')^2 &< 2(a')(b')^2 + 2b' \\
a'\big((a')^2 + (b')^2\big) &< 2b'(a'b' + 1) \\
a'\left(\frac{(a')^2 + (b')^2}{a'b' + 1}\right) &< 2b' \\
ka' &< b' + b' \\
ka' - b' &< b' \\
b^* &< b'
\end{aligned}$$

Therefore, we proved that $b^* < b'$.

Now, we are given that (a', b') is a solution to $k = \frac{x^2+y^2}{xy+1}$, where $b' \geq a'$. So, we have

$$k = \frac{a'^2 + b'^2}{a'b' + 1} \implies (a')^2 - kb'a' + (b')^2 - k = 0$$

By trivial observation, we can notice that the above equation is a quadratic equation, so we know that there exists two solutions. Also, we know that that b' is a solution to

$$(a')^2 - ka'y + y^2 - k = 0 \tag{2}$$

Let b'_1 be the other solution that satisfy $(a')^2 - ka'y + y^2 - k = 0$, such that $(a')^2 - ka'(b'_1) + (b'_1)^2 - k = 0$. Note that we may represent equation (3), as $(y - b')(y - b'_1) = 0$ implying

$$(a')^2 - (b' + b'_1)y + y^2 - k = 0$$

by comparing we can claim that $(b' + b'_1) = ka'$. So, we get

$$(b' + b'_1) = ka' \implies b'_1 = ka' - b'$$

We know that $b^* = ka' - b'$ and we showed that $b'_1 = ka' - b'$, therefore $b^* = b'$. Hence, we proved that (a', b^*) is a solution.

We will now prove that $b^* > 0$. We showed that (a', b^*) is a solution to $k = \frac{x^2+y^2}{xy+1}$. So, we get

$$k = \frac{(a')^2 + (b^*)^2}{(a')(b^*) + 1} = k$$

We know that $k > 0$. We will claim that $(a')^2 + (b^*)^2 > 0$, since it is sum of two squares. we know $1 > 0$ and we were given that $a' \geq 1 > 0$. If $b^* \leq 1$, then $(a')(b^*) + 1 < 0$, which implies that $k < 0$, which contradicts the fact that $k > 0$. Therefore, $b^* \geq 0$ implying that b^* cannot be negative. Hence proving our claim.

We will now prove that k must be a perfect square. For the sake of contradiction, we will assume that k is not a perfect square. We are given that positive integers a' and b', with $b' \geq a'$ are the minimal solution to

$$k = \frac{x^2 + y^2}{xy + 1}$$

where the minimal is among all values of $x + y$. By part (1), we proved that $b^* < b'$, where $b^* = ka' - b'$, and also by part (2), we know that (a', b^*) is a solution to $k = \frac{x^2+y^2}{xy+1}$, and by part (3) we know that $b^* > 0$. So, we can observe that the positive pair, $(a', b^*) < (a', b')$, i.e., (a', b^*) is more smaller (minimal) integer solution to $k = \frac{x^2+y^2}{xy+1}$. Hence a contradiction, since we assumed (a', b') is the minimal positive integer solution, also since we know my part (1) that $b^* < b'$, we can $b^* \neq b'$. Therefore, we proved by contradiction that k must be a perfect square. SO DONE! ॐ