These are problems proposed by the TAs of the MSRI Summer Graduate School on Automorphic Forms, Summer 2017, in Berkeley.

# 1 Day 1

**1** Look up definition of inverse limits (of topological groups). Understand why

$$\mathrm{Gal}(L/K) = \varprojlim_{L \supseteq M \supseteq K} \mathrm{Gal}(M/K),$$

as topological groups.

**Solution** Recall that the inverse limit of topological groups $G_i$ with respect to some inverse system $\{\varphi_{ij}\}_{i,j \in I, i \leq j}$ is the unique (up to unique isomorphism) topological group $G$ with the property that any topological $H$ admitting continuous homomorphisms $\{\psi_i\}_{i \in I}$ to the $G_i$ which are compatible with the $\{\varphi_{ij}\}$ factors uniquely through $G$. $\mathrm{Gal}(L/K)$ admits natural maps to all finite $\mathrm{Gal}(L_i/K)$. Suppose $H$ is a topological group admitting maps $\{\psi_i \colon H \to \mathrm{Gal}(L_i/K)\}_{i \in I}$, compatible with the inverse system $\{\varphi_{ij} \colon \mathrm{Gal}(L_i/K) \to \mathrm{Gal}(L_j/K)\}_{L_i \supseteq L_j}$. Then there exists a unique continuous map $\psi \colon H \to \mathrm{Gal}(L/K)$ which is compatible with everything, defined as follows: let $\psi(h) =: g$ for $h \in H$ be the element $g \in \mathrm{Gal}(L/K)$ such that $g(\lambda) = \psi_i(h)(\lambda)$, for some $i$ such that $h \in L_i$, and for all $\lambda \in L_i$. If this map is well-defined, it is manifestly unique. Suppose $\lambda \in L_1, L_2$, and let $L_3$ be the compositum of $L_1$ and $L_2$. Then $\psi_i(h)(\lambda) = \varphi_{31}\psi_3(h)(\lambda) = \varphi_{32}\psi_3(h)(\lambda)$. Also, the compatibility of the $\{\varphi_{ij}\}$ with the $\{\psi_i\}$ ensures that $\psi(h)$ is an honest element of $\mathrm{Gal}(L/K)$. Thus $\psi$ exists and is unique.

**2** View

$$\mathrm{Gal}(L/K) \hookrightarrow \prod_{L \supseteq M \supseteq K} \mathrm{Gal}(M/K).$$

Prove that $\mathrm{Gal}(L/K)$ is closed when viewed as a subspace of the product of discrete finite groups (equipped with product topology).

**Solution** As before, let $I$ be an index for all finite Galois $L_i/K$. Suppose $g = (g_i)_{i \in I} \notin \mathrm{Gal}(L/K)$. This means that there is some $L_1 \supseteq L_0$ such that $\mathrm{res}_{L_0} g_1 \neq g_0$ (here, $\mathrm{res}_{L_0}$ denotes the restriction map $\mathrm{res}_{L_0} \colon \mathrm{Gal}(L_1/K) \to \mathrm{Gal}(L_0/K)$. Also, in using numerical indices, I don't mean to hint at any filtration of the fields, I just didn't want to keep using indices i,j). Then $U = \{(g_0, g_1)\} \times \prod_{i \neq 0,1} \mathrm{Gal}(L_i/K)$ is a basic open set containing $g$ which does not intersect $\mathrm{Gal}(L/K)$. Thus, the complement of $\mathrm{Gal}(L/K)$ is open in the product topology.

**3** Understand details of Example 0: $K = \mathbf{Q}$, $L_n = \mathbf{Q}(\zeta_{p^n})$, $L = \bigcup_{n \geq 1} L_n \implies \mathrm{Gal}(L/K) = \mathbf{Z}_p^\times$. From lecture,

$$\mathrm{Gal}(L/K) \hookrightarrow \prod_{n \geq 1} (\mathbf{Z}/p^n\mathbf{Z})^\times$$
$$\varphi \mapsto (\varphi_n)_{n \geq 1}.$$

- a) Compare $\prod_{n \geq 1} (\mathbf{Z}/p^n\mathbf{Z})^\times (= \prod_n \mathrm{Gal}(L_n/K))$ with $\prod_{L \supseteq M \supseteq K} \mathrm{Gal}(M/K)$. Why is the first product enough to compute $\mathrm{Gal}(L/K)$?
- b) Prove $\varphi_m \equiv \varphi_n \mod p^m$ $(m \leq n)$

**Solution** For part (a), note that there is a natural, continuous surjection

$$\prod_{L\supseteq M\supseteq K} \mathrm{Gal}(M/K) \twoheadrightarrow \prod_n \mathrm{Gal}(L_n/K)$$

which is given by projection onto the factors of the form $\mathrm{Gal}(L_n/K)$. Now one has to check that the composition

$$\mathrm{Gal}(L/K) \hookrightarrow \prod_{L\supseteq M\supseteq K} \mathrm{Gal}(M/K) \twoheadrightarrow \prod_n \mathrm{Gal}(L_n/K)$$

is injective. An element of $\mathrm{Gal}(L/K)$ is determined by its action on each $\mathrm{Gal}(L_i/K)$ (as every $\lambda \in L$ is contained in some $L_i$), hence the desired injectivity. So we can compute $\mathrm{Gal}(L/K)$ as topological subgroup of $\prod_n \mathrm{Gal}(L_n/K)$.

For b, observe that if $\sigma \in \mathrm{Gal}(L_n/K)$, is given by $[\zeta_{p^n} \mapsto \zeta_{p^n}^a]$ (so that $\sigma$ is identified with $a \in (\mathbf{Z}/p^n\mathbf{Z})^\times$), then the image of $\sigma$ under the natural map $\mathrm{Gal}(L_n/K) \to \mathrm{Gal}(L_m/K)$ for $m \neq n$ is $[\zeta_{p^m} \to \zeta_{p^m}^a]$ where now we view $a$ as an element of $(\mathbf{Z}/p^m\mathbf{Z})^\times$ via the natural reduction $\mathbf{Z}/p^n\mathbf{Z} \to \mathbf{Z}/p^m\mathbf{Z}$. This is because $\zeta_{p^m} = \zeta_{p^n}^{p^{n-m}}$, so $\sigma(\zeta_{p^m}) = \zeta_{p^n}^{ap^{n-m}} = \zeta_{p^m}^a$

**4** Given a topological ring $R$, we ask whether the inversion map $i\colon R^\times \to R^\times$, $r \mapsto r^{-1}$, is continuous when $R^\times$ is given the subspace topology from $R$.

- a) Show $i$ is invertible when $R = \mathbf{Z}_p$.
- b) Given an example of a topological ring $R$ such that $i$ is not continuous.

**Solution** For part (a), let $x, y \in \mathbf{Z}_p^\times$, $|x-y| < \epsilon$. Then

$$|\frac{1}{x} - \frac{1}{y}| = |\frac{y-x}{xy}| < \epsilon/|xy| < \epsilon.$$

For part (b), let $R = \mathbf{Q}$, and let $\{x + n\mathbf{Z}\}$ be a basis of open sets for all $n \in \mathbf{Z}_{\geq 1}$ and $x \in \mathbf{Q}$. $\mathbf{Q}^\times$ is open in this topology. Let $U$ be the open set $\mathbf{Z}\cap$. The preimage of this set is $\{\pm 1, \pm 1/2, \pm 1/3, \cdots\}$, but this set cannot be open as it contains no sets of the form $x + n\mathbf{Z}$.

**5** Check that $\mathrm{Gal}(\mathbf{F}_{q^n}/\mathbf{F}_q) = \mathbf{Z}/n\mathbf{Z}$, via $\mathrm{Frob}_q \mapsto 1$, and that $\mathrm{Frob}_q$ has order $n$ (recall in this course that = means canonically isomorphic).

**Solution** We must show that $\mathrm{Frob}_q$ has order $n$. Since $\#\mathrm{Gal}(\mathbf{F}_{q^n}/\mathbf{F}_q) = n$, this will prove the isomorphism.

Suppose $[x \mapsto x^{q^m}]$ is the identity for some $m \geq 1, m \mid n$. But then every element of $\mathbf{F}_{q^n}$ is fixed by the $q^m$-th power map, which contradicts the fact that $\mathbf{F}_{q^n}^\times$ is cyclic.

**6** Let $K = \mathbf{F}_q$ and fix an algebraic closure $\overline{K}$. We have that $\Phi\colon \mathrm{Gal}(\overline{K}/K) \hookrightarrow \prod_{n\geq 1} \mathbf{Z}/n\mathbf{Z}$, $g \mapsto (g_n)_{n\geq 1}$. Prove

$$\mathrm{Im}(\Phi) = \{(a_n)_{n\geq 1} \in \prod_{n \geq 1} \mathbf{Z}/n\mathbf{Z} \colon \forall m \mid n, a_n \equiv a_m \mod m\}.$$

**Solution** Suppose $\sigma \in \mathrm{Gal}(\overline{K}/K)$. If $\sigma$ acts by $a_n \in \mathbf{Z}/n\mathbf{Z}$ on $\mathrm{Gal}/(\mathbf{F}_{q^n}/K)$, that is, if $\sigma|_{\mathbf{F}_{q^n}} : x \mapsto x^{q^{a_n}}$, then $\sigma|_{\mathbf{F}_{q^m}} : x \mapsto x^{q^{a_n}}$, where $a_n$ is viewed $\mod m$. Thus $a_m \equiv a_n \mod m$. This shows the image of $\Phi$ is indeed the set subgroup of $\prod_n \mathbf{Z}/n\mathbf{Z}$ above.

**7** Give an example of an algebraic extension $L/\mathbf{Q}_p$ such that $\mathfrak{p}_L$ is not principal (same question if you replace $\mathbf{Q}_p$ with a finite extension).

**Solution** Let $L = \overline{\mathbf{Q}_p}$. Suppose for the sake of contradiction that $\mathfrak{p}_L$ is principal, with normalized valuation $v = v_L$. Then there is a $\pi_L$ with $v(\pi_L) = 1$. But $\sqrt{\pi_L} \in \overline{L}$, with $v(\pi_L) = 1/2$. Contradiction.

**8** Let $K/\mathbf{Q}_p$ be a finite extension with $L/K$ Galois.

- a) Prove that $\mathrm{Gal}(L/K) \to \mathrm{Gal}(k_L/k_K)$ is surjective.
- b) Give an example of $L/K$ where the above map is NOT injective (also, one where it *is* injective).

**Solution** (a) Actually quite annoying given how we defined "unramified" (our definition of unramified in lecture assumed the map given was surjective). The idea of the solution is to lift roots of a miminal polynomial for generator of the residue field and use the fact that Galois is transitive on roots of an irreducible factor to show the surjectivity. Carefully pass to the limit to get the result for infinite extensions by doing this lifting compatibly in towers.

(b) A ramified extension of fields is $\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$. An unramified extension is $\mathrm{Gal}(\mathbf{Q}_p(\zeta_n)/\mathbf{Q}_p)$ for $n$ prime to $p$.

**9** Let $K/\mathbf{Q}_p$ be finite, $\pi_K$ a uniformizer for $K$. Prove that the following are equivalent:

- a) $\mathfrak{p}_L = \pi_K \mathcal{O}_L$
- b) $v_K(L^\times) = \mathbf{Z}$
- c) $L/K$ is unramified (I assume Galois as well, since we only defined unramified for Galois extensions).

**Solution** Assume (a). Then every nonzero element of $L$ can be written as an integer power of $\pi_K$ times a unit in $\mathcal{O}_L^\times$, and $v(\ell) = 0$ for any unit $\ell \in \mathcal{O}_L^\times$. So all elements of $L$ have integral valuation under $v_K$.

Now assume (b). We need to show that $\mathrm{Gal}(L/K) \to \mathrm{Gal}(k_L/k_K)$ is an isomorphism, or, equivalently, that it's injective. It's enough to assume $L/K$ is finite, and then pass to a limit. Then $n = [L\colon K] = ef$, where $f$ is residue field degree and where $e$ is such that $v_L = ev_K$, where $v_L$ is the normalized valuation of $L$. But $e = 1$ in our case, so $n = f$. This proves injectivity in the finite case.

Finally assume c, and again assume for simplicity that $L/K$ is finite. If $\mathfrak{p}_L = \pi_K^e \mathcal{O}_L$ for some $e > 1$, then $v_L = ev_K$, but this would mean that $f < n$ by the argument above, a contradiction. So we have $e = 1$.

**10** The compositum of two unramified extensions (of $K/\mathbf{Q}_p$ finite) is again unramified.

**Solution** Let $L/K$, $L'/K$ be two unramified extensions. Then as every element of $K_1K_2$ is a finite sum or product of elements of $K_1K_2$, we still have $v_K((LL')^\times) = \mathbf{Z}$, thus the compositum is unramified.

**11** Let $L/K$ be an algebraic extension.

- a) There is a unique maximal unramified extension $L \supseteq M \supseteq K$.
- b) $\mathrm{Gal}(M/K) \simeq \mathrm{Gal}(k_M/k_K)$ is (pro-)cyclic.

**Solution** For part (a), Zorn's lemma (all chains have upper bound in collection of algebraic extensions, namely the algebraic closure) guarantees that maximal unramified extensions exist, and by the previous exercise, there can be only one such maximal unramified extension.

For part b, we know that Galois groups of finite fields are (pro)-cyclic.

# 2 Day 2

In the following exercises, $K/\mathbf{Q}_p$ is finite and $L/K$ is Galois.

**1** Show that

$$K^{nr} = \bigcup_{m \geq 1, p \nmid m} K(\zeta_m)$$

where $\zeta_m$ denotes a primitive $m$-th root of unity. Hint: Consider extensions of $k_K$.

**2** If $\sigma \in \mathrm{Gal}(L/K)$, then $\sigma(\mathcal{O}_L) = \mathcal{O}_L$, $\sigma(\mathfrak{p}_L) = \mathfrak{p}_L$.

**3** Assume $I_{L/K}$ is finite. Show the following.

- a) For all $i \geq 0$, $I_{L/K,i}$ is a normal subgroup of $I_{L/K}$.
- b) For all $i \geq 0$, $I_{L/K,i} \supseteq I_{L/K,i+1}$.
- c) $I_{L/K,i} = \{1\}$ for $i$ sufficiently large.
- d) $I_{L/K,i}/I_{L/K,i+1} \cong (\mathbf{Z}/p\mathbf{Z})^{n_i}$ (some $n_i \in \mathbf{Z}_{\geq 0}$), for all $i \geq 1$.

**4** Give an example of $L'/L/K$ such that $I_{L/K,i}$ is NOT the image of $I_{L'/K,i}$ for some $i$ under the natural map $I_{L'/K} \to I_{L/K}$.

**5** Read Proposition IV.3.14 (or all of IV.3) in Serre's Local Fields. This is the discussion of the upper numbering and its behavior under extensions of fields.

**6a** Show that the jumps of the upper numbering occur at rational numbers

**6b** (See Serre, Exercise IV.3.2) But jumps don't have to be at integers! Let $G$ be the group of quaternions ($\{\pm 1, \pm i, \pm j, \pm k\}$ subject to relations $i^2 = j^2 = k^2 = -1f, ijk = -1$). Suppose that $G$ is the Galois group of a totally ramified extension. Show

- $G = G_0 = G_1$.
- $\{\pm 1\} = G_2 = G_3$.
- $G^v = G$ for all $v \leq 1$.
- $G^v = \{\pm 1\}$ for $1 < v \leq 3/2$.
- $G^v = \{1\}, v > 3/2$.

**7** The compositum of two tamely ramified (resp, abelian) is tamely ramified (resp, abelian).

**8** Composition of abelian extensions need not be abelian: $K = \mathbf{Q}_3$, let $L$ be the Galois closure of $K(\sqrt[4]{2})$. Show that $\mathrm{Gal}(L/K)$ is not abelian, but that there exists $L \supseteq M \supseteq K$ such that $M/K$ is Galois and $\mathrm{Gal}(L/M)$ and $\mathrm{Gal}(M/K)$ are abelian.

**9** Recall that $K^t$ denotes the maximal tamely ramified extension, and $K^{nr}$ the maximal unramified extension of $K$.

- a) $K^t = \cup_{p\nmid m, m\geq 1} K^{nr}(\sqrt[m]{\pi_K})$. Hint: Kummer theory.
- b) Show that

$$\mathrm{Gal}(K^{nr}(\sqrt[m]{\pi_K}/K^{nr})/K^{nr}) \to \mu_m$$
$$\sigma \mapsto \frac{\sigma(\sqrt[m]{\pi_K})}{\sqrt[m]{\pi_K}}$$

  is well-defined and an isomorphism.

**10** Recall that $\mathrm{Gal}(K^t/K^{nr}) = \varprojlim_{p\nmid m} \mu_m$. Check that if $\sigma \longleftrightarrow \zeta$ under this isomorphism, then $\tilde{\mathrm{Frob}} \circ \sigma \circ \tilde{\mathrm{Frob}}^{-1} \longleftrightarrow \zeta^q$, where $q = \#k_K$.

**11** Show that the following sequence of topological groups splits.

$$1 \to \mathrm{Gal}(K^t/K^{nr}) \to \mathrm{Gal}(K^t/K) \to \mathrm{Gal}(K^{nr}/K) \to 1.$$

Hint: choosing a lift $\tilde{\mathrm{Frob}} \in \mathrm{Gal}(K^t/K)$ of Frob gives rise to a homomorphism $\mathbf{Z} \to \mathrm{Gal}(K^t/K)$. Show it extends to all of $\hat{\mathbf{Z}}$ (continuously) (recall that $\hat{\mathbf{Z}} = \mathrm{Gal}(K^{nr}/K)$).

**12** Let $G$ be an arbitrary topological group (so multiplication and inversion are continuous.

- a) Show that the closure of the identity is a normal subgroup of $G$. Call it $H$.
- b) Show that $G/H$ is a Hausdorff topological group.